

AEROSPACE STRUCTURAL ANALYSIS

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Chapter 1

Basic Equations of Linear Elasticity

1.1 The concept of stress

1.1.1 The state of stress at a point

The state of stress in a solid body can be visualized by cutting the solid by a plane. The effect of the removed portion of the body is represented by forces acting on the remaining portion. Fig. 1.1 shows the body cut by a plane normal to the direction defined by the unit vector $\mathbf{n}$. Consider now a small area da_n located in this plane at point P . Let $\mathbf{F}_n$ be the force vector acting on that small element of area; this vector has units of forces, i.e. N . The *stress vector* τ_n acting at that point is defined as

$$\tau_n = \lim_{da_n \rightarrow 0} \left(\frac{\mathbf{F}_n}{da_n} \right). \quad (1.1)$$

The existence of the stress vector, i.e. the existence of this limit is a basic assumption of the theory of elasticity. The stress vector has units of force per unit area, i.e. Pa . The force and stress vectors are assumed to be applied at the centroid of the element of area on which they act. The total force acting on the differential element is

$$\mathbf{F}_n = da_n \tau_n. \quad (1.2)$$

Of course, the force and stress vectors depend on the orientation of the element on which they act. A more detailed representation of the state of stress at point P can be visualized by cutting a differential element of volume around that point. Fig. 1.2 shows this differential element of volume bounded by faces normal to the axes $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ of a Cartesian system $\mathcal{S}$. Let τ_1 , τ_2 , and τ_3 be the stress vectors acting on the faces normal to $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, respectively. The stress vector acting on each face is resolved along the reference frame $\mathcal{S}$ as

$$\begin{aligned} \tau_1 &= \sigma_1 \mathbf{i}_1 + \tau_{12} \mathbf{i}_2 + \tau_{13} \mathbf{i}_3; \\ \tau_2 &= \tau_{21} \mathbf{i}_1 + \sigma_2 \mathbf{i}_2 + \tau_{23} \mathbf{i}_3; \\ \tau_3 &= \tau_{31} \mathbf{i}_1 + \tau_{32} \mathbf{i}_2 + \sigma_3 \mathbf{i}_3. \end{aligned} \quad (1.3)$$

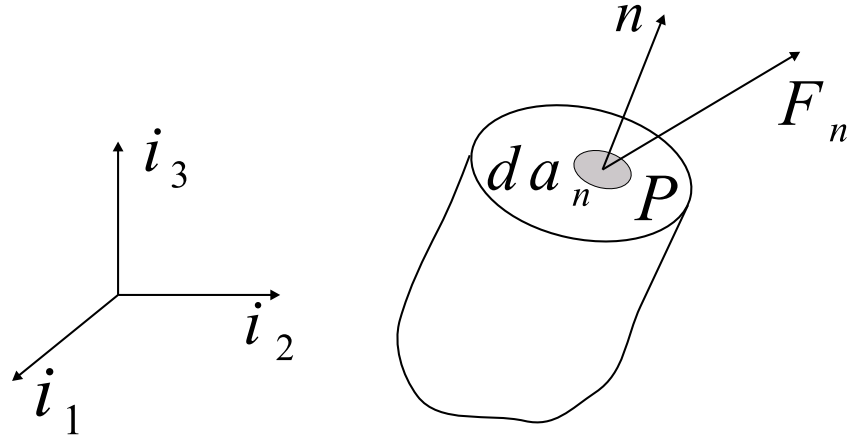
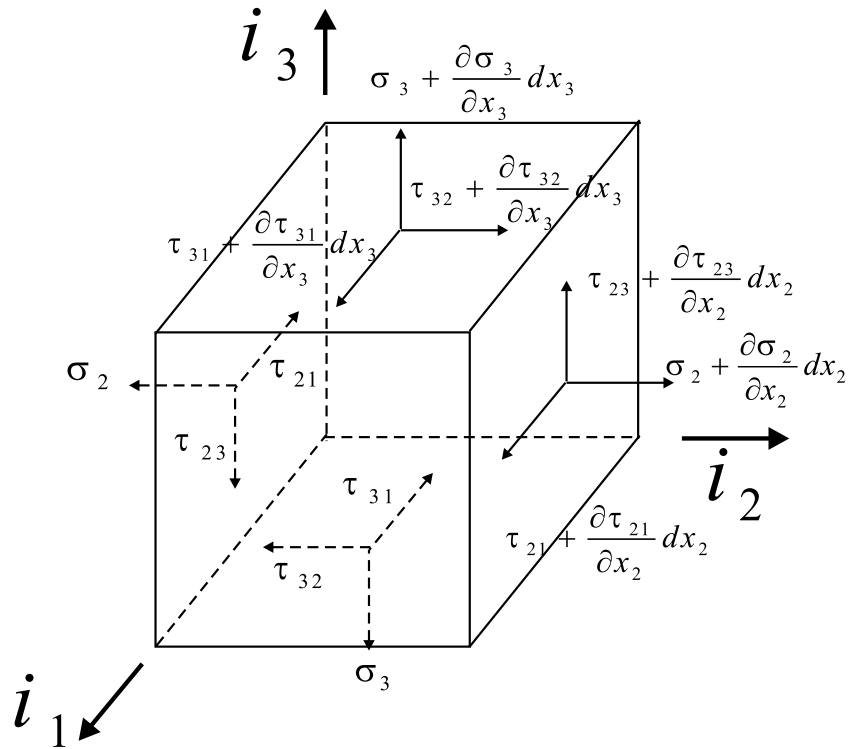


Figure 1.1: A solid body cut by a plane.

Figure 1.2: Stress components acting on a differential element of volume. For clarity of the figure, the stress components acting on the faces normal to $\mathbf{i}_1$ were not drawn.

The various stress components $\sigma_1, \tau_{12}, \tau_{13}$, etc. are called the *engineering stress components*. The stress components σ_1, σ_2 , and σ_3 are normal to the faces on which they act, and are called *direct*, or *normal stresses* whereas the stress components $\tau_{23}, \tau_{13}, \tau_{12}$, etc. are parallel to the faces on which they act and are called *shearing stresses*.

Of course the stresses acting on two parallel faces are not equal. Consider, for instance, the two faces normal to $\mathbf{i}_1$ and passing through points P and R . Since the distance dx_1 between these two faces is infinitesimal, the corresponding stresses are σ_1 and $\sigma_1 + \partial\sigma_1/\partial x_1 dx_1$, respectively. Similarly, the shearing stresses acting on those same faces are τ_{12} and τ_{13} , and $\tau_{12} + \partial\tau_{12}/\partial x_1 dx_1$ and $\tau_{13} + \partial\tau_{13}/\partial x_1 dx_1$, respectively.

1.1.2 Volume equilibrium equations

The differential element of volume depicted in fig. 1.2 must be in equilibrium under the effect of the stresses acting on its six faces and body forces per unit volume $\mathbf{b}$ acting at its centroid. The components of this body force vector resolved in $\mathcal{S}$ are $\mathbf{b} = b_1 \mathbf{i}_1 + b_2 \mathbf{i}_2 + b_3 \mathbf{i}_3$. These body forces could be gravity forces, inertial forces, or forces of an electric or magnetic origin.

According to Newton's law, the sum of all the forces acting on this differential element should vanish. Considering all the forces acting along the $\mathbf{i}_1$ direction, yields

$$\begin{aligned} -\sigma_1 dx_2 dx_3 + \left(\sigma_1 + \frac{\partial \sigma_1}{\partial x_1} dx_1 \right) dx_2 dx_3 - \tau_{21} dx_1 dx_3 + \left(\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2} dx_2 \right) dx_1 dx_3 \\ - \tau_{31} dx_1 dx_2 + \left(\tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} dx_3 \right) dx_1 dx_2 + b_1 dx_1 dx_2 dx_3 = 0 \end{aligned} \quad (1.4)$$

After simplification, the equilibrium condition becomes

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 = 0. \quad (1.5)$$

The forces along the axes $\mathbf{i}_2$ and $\mathbf{i}_3$ must vanish as well, and a similar reasoning yields the following equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 &= 0; \\ \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 &= 0; \\ \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 &= 0, \end{aligned} \quad (1.6)$$

which must be satisfied at all points inside the body.

To satisfy all equilibrium requirements, the sum of all the moments acting on the differential element of volume depicted in fig. 1.2 must also vanish. Starting with all the moments about axis $\mathbf{i}_1$ yields

$$\begin{aligned} \tau_{23} dx_1 dx_3 \frac{dx_2}{2} + \left(\tau_{23} + \frac{\partial \tau_{23}}{\partial x_2} dx_2 \right) dx_1 dx_3 \frac{dx_2}{2} \\ - \tau_{32} dx_1 dx_3 \frac{dx_3}{2} - \left(\tau_{32} + \frac{\partial \tau_{32}}{\partial x_3} dx_3 \right) dx_1 dx_2 \frac{dx_3}{2} = 0. \end{aligned} \quad (1.7)$$

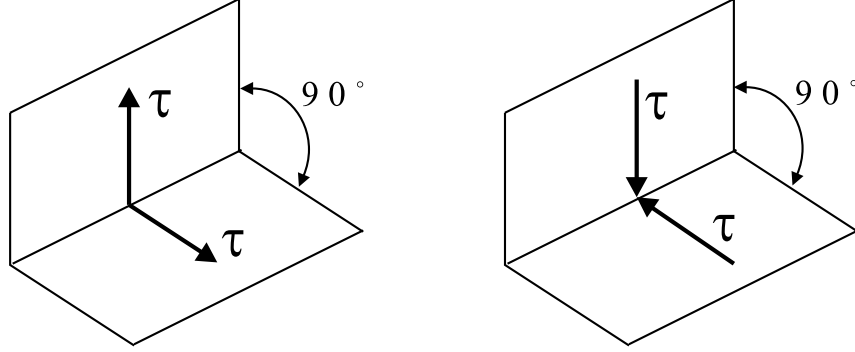


Figure 1.3: Reciprocity of the shearing stresses acting on two orthogonal faces.

Neglecting higher order terms and simplifying reduces this equilibrium condition to

$$\tau_{23} - \tau_{32} = 0. \quad (1.8)$$

Enforcing the vanishing of the sum of the moments about axes $\mathbf{i}_2$ and $\mathbf{i}_3$ leads to similar equations

$$\tau_{23} = \tau_{32}; \quad \tau_{13} = \tau_{31}; \quad \tau_{12} = \tau_{21}. \quad (1.9)$$

The implication of these equalities are summarized by the principle of reciprocity of shearing stresses: *"The shearing stresses acting in the direction normal to the common edge of two orthogonal faces must be equal in magnitude and be simultaneously oriented toward or away from the common edge."* Fig. 1.3 depicts this reciprocity principle.

1.1.3 Surface equilibrium equations

At the outer surface of the body, the internal stresses must be in equilibrium with the externally applied surface tractions. The applied surface traction vector $\hat{\mathbf{t}}$ is resolved in the reference frame $\mathcal{S}$ as $\hat{\mathbf{t}} = \hat{t}_1 \mathbf{i}_1 + \hat{t}_2 \mathbf{i}_2 + \hat{t}_3 \mathbf{i}_3$. Fig. 1.4 shows a tetrahedron bounded by three faces normal to axes $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, and by a fourth face which is a differential element of area da_n of the outer surface of the body. The unit normal to this element of area is denoted $\mathbf{n}$ with components $\mathbf{n} = n_1 \mathbf{i}_1 + n_2 \mathbf{i}_2 + n_3 \mathbf{i}_3$.

Equilibrating the forces acting along the axis $\mathbf{i}_1$ implies

$$\hat{t}_1 da_n - \sigma_1 \frac{dx_2 dx_3}{2} - \tau_{21} \frac{dx_1 dx_3}{2} - \tau_{31} \frac{dx_1 dx_2}{2} = 0. \quad (1.10)$$

Dividing this equilibrium condition by the area da_n yields

$$\hat{t}_1 = \sigma_1 \frac{dx_2 dx_3}{2da_n} + \tau_{21} \frac{dx_1 dx_3}{2da_n} + \tau_{31} \frac{dx_1 dx_2}{2da_n}. \quad (1.11)$$

Inspecting fig. 1.4 shows that

$$n_1 = \frac{dx_2 dx_3}{2da_n}; \quad n_2 = \frac{dx_1 dx_3}{2da_n}; \quad n_3 = \frac{dx_1 dx_2}{2da_n}, \quad (1.12)$$

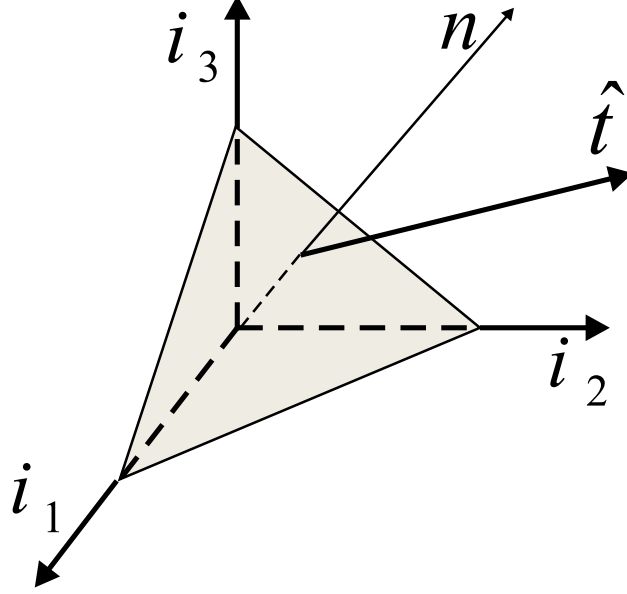


Figure 1.4: A tetrahedron with one face along the outer surface of the body.

and hence $\hat{t}_1 = n_1 \sigma_1 + n_2 \tau_{21} + n_3 \tau_{31}$. It is convenient to define the surface traction $t_1 = n_1 \sigma_1 + n_2 \tau_{21} + n_3 \tau_{31}$, and the surface equilibrium equation then simply writes $\hat{t}_1 = t_1$, *i.e.* the surface traction is equal to the prescribed surface traction. Considering the sum of the forces acting along the axes $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ yields the surface equilibrium equations

$$t_1 = \hat{t}_1; \quad t_2 = \hat{t}_2; \quad t_3 = \hat{t}_3, \quad (1.13)$$

where the surface tractions were defined as

$$\begin{aligned} t_1 &= n_1 \sigma_1 + n_2 \tau_{21} + n_3 \tau_{31}; \\ t_2 &= n_1 \tau_{12} + n_2 \sigma_2 + n_3 \tau_{32}; \\ t_3 &= n_1 \tau_{13} + n_2 \tau_{23} + n_3 \sigma_3. \end{aligned} \quad (1.14)$$

A body is said to be in equilibrium if eqs. (1.7) are satisfied at all points inside the body, and eqs. (1.13) at all points of the external surface of the body.

It is often important to consider the component of surface traction t_n normal to the outer surface of the body, where $t_n = \mathbf{t} \cdot \mathbf{n}$. This surface traction is found from eq. (1.14) as

$$t_n = n_1^2 \sigma_1 + n_2^2 \sigma_2 + n_3^2 \sigma_3 + 2n_2 n_3 \tau_{23} + 2n_1 n_3 \tau_{13} + 2n_1 n_2 \tau_{12}. \quad (1.15)$$

The surface traction vector can then be resolved into its normal component and a component $\mathbf{t}_s$ acting in the plane of the external surface, *i.e.* $\mathbf{t} = t_n \mathbf{n} + \mathbf{t}_s$. The in-plane component is

$$\mathbf{t}_s = \mathbf{t} - t_n \mathbf{n} = (t_1 - n_1 t_n) \mathbf{i}_1 + (t_2 - n_2 t_n) \mathbf{i}_2 + (t_3 - n_3 t_n) \mathbf{i}_3. \quad (1.16)$$

1.1.4 The plane state of stress

A particular state of stress of great partial importance is the *plane state of stress*. In this case, the only non vanishing stress components are σ_1 , σ_2 , and τ_{12} , and furthermore, these stress components are assumed to be independent of x_3 . This state of stress occurs, for instance, in a very thin plate subjected to loads applied in its own plane.

The equations of equilibrium (1.7) considerably simplify in the plane stress case

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + b_1 = 0; \quad \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + b_2 = 0. \quad (1.17)$$

Similar simplifications take place for the definition of the surface tractions eqs. (1.14)

$$t_1 = n_1 \sigma_1 + n_2 \tau_{21}; \quad t_2 = n_1 \tau_{12} + n_2 \sigma_2, \quad (1.18)$$

and the surface equilibrium equations reduce to $t_1 = \hat{t}_1$ and $t_2 = \hat{t}_2$. For this two dimensional problem, the components of the normal vector are easily expressed in terms of the angle θ between axis $\mathbf{i}_1$ and the normal

$$n_1 = \cos \theta; \quad n_2 = \sin \theta. \quad (1.19)$$

The normal component of surface traction (1.15) becomes

$$t_n = \cos^2 \theta \sigma_1 + \sin^2 \theta \sigma_2 + 2 \sin \theta \cos \theta \tau_{12}, \quad (1.20)$$

and the in-plane component of the surface traction (1.16) is

$$t_s = \sin \theta \cos \theta (\sigma_2 - \sigma_1) + (\cos^2 \theta - \sin^2 \theta) \tau_{12} + 2\tau_{12}. \quad (1.21)$$

1.2 Analysis of the state of stress at a point

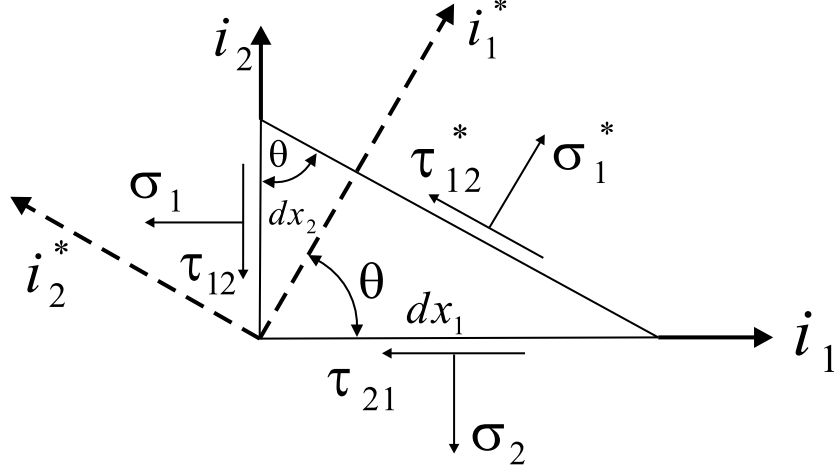
The state of stress at a point has been characterized in the previous section by the axial and shearing stresses acting on the faces of a differential element of volume cut in the solid. The faces of this cube were cut normal to the axes of a Cartesian reference frame $\mathcal{S}$, and the stress vector acting on these faces were resolved along the same axes. It is clear that the orientation of this reference frame is entirely arbitrary. A reference frame $\mathcal{S}^*$ with unit vectors $\mathbf{i}_1^*$, $\mathbf{i}_2^*$, and $\mathbf{i}_3^*$ could have been selected, and an analysis identical to that of the previous section would have led to the definition of axial stresses σ_1^* , σ_2^* , σ_3^* , and shearing stresses τ_{23}^* , τ_{13}^* , τ_{12}^* , etc. A typical equilibrium equation at a point of the body would write

$$\frac{\partial \sigma_1^*}{\partial x_1^*} + \frac{\partial \tau_{21}^*}{\partial x_2^*} + \frac{\partial \tau_{31}^*}{\partial x_3^*} + b_1^* = 0. \quad (1.22)$$

where the notation $(.)^*$ is used to indicate the components of the corresponding quantity measured in $\mathcal{S}^*$. A typical surface traction would be defined as

$$t_1^* = n_1^* \sigma_1^* + n_2^* \tau_{21}^* + n_3^* \tau_{31}^*. \quad (1.23)$$

Though expressed in different reference frames, eqs. (1.7) and (1.22) or (1.14) and (1.23) express the same equilibrium conditions for the body. Therefore, the stresses resolved in the two reference frames should be closely related.

Figure 1.5: Differential element with a face at an angle θ .

1.2.1 Rotation of stresses

We propose to relate the components of stresses in two different Cartesian systems $\mathcal{S}$ and $\mathcal{S}^*$. To simplify the derivation, a two dimensional case is considered first.

Consider the differential element of triangular shape depicted in fig. 1.5. Two sides of the triangle are aligned with the axes $\mathbf{i}_1$ and $\mathbf{i}_2$, and the last side is at an angle θ . The stress components acting on the various face of the triangle are also depicted on the free body diagram. Summing up all the forces in the $\mathbf{i}_1^*$ direction gives:

$$ds \sigma_1^* = dx_1(\sigma_2 \sin \theta + \tau_{12} \cos \theta) + dx_2(\sigma_1 \cos \theta + \tau_{12} \sin \theta), \quad (1.24)$$

where ds is the length of the face at an angle θ . Dividing this relationship by ds , and noting that $dx_1/ds = \sin \theta$ and $dx_2/ds = \cos \theta$, yields:

$$\sigma_1^* = \cos^2 \theta \sigma_1 + \sin^2 \theta \sigma_2 + 2 \sin \theta \cos \theta \tau_{12}. \quad (1.25)$$

A summation of the forces in the $\mathbf{i}_2^*$ direction gives:

$$\tau_{12}^* = -\sin \theta \cos \theta \sigma_1 + \sin \theta \cos \theta \sigma_2 + (\cos^2 \theta - \sin^2 \theta) \tau_{12}. \quad (1.26)$$

The stress component σ_2^* can be found by replacing θ by $\theta + \pi/2$ in (1.25). In summary, these results can be written in a matrix form as:

$$\begin{bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}, \quad (1.27)$$

where $m = \cos \theta$ and $n = \sin \theta$.

This relation can be readily inverted to yield:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{bmatrix}. \quad (1.28)$$

With the help of basic trigonometric identities, the transformation rules for stress components (1.27) write

$$\begin{aligned}\sigma_1^* &= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau_{12} \sin 2\theta; \\ \sigma_2^* &= \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta - \tau_{12} \sin 2\theta; \\ \tau_{12}^* &= -\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta + \tau_{12} \cos 2\theta.\end{aligned}\tag{1.29}$$

This important result shows that the knowledge of the stress components σ_1 , σ_2 , and τ_{12} on two orthogonal faces allows the computation of the stress components acting on a face with an arbitrary orientation. In other words, the knowledge of the stress components on two orthogonal faces fully defines the state of stress at that point.

Adding the axial stresses on two orthogonal faces yields the following result

$$\sigma_1 + \sigma_2 = \sigma_1^* + \sigma_2^*;\tag{1.30}$$

which means that the sum of the axial stresses acting on two orthogonal faces is a constant for all orientations of these faces.

1.2.2 Principal stresses

The axial stress acting on a face can be viewed as a function of θ , the orientation angle for this face. It is interesting to find the orientation θ_p corresponding to the maximum or minimum value of that stress. To that effect, the derivative of σ_1^* with respect to θ is set to zero

$$\frac{d\sigma_1^*}{d\theta} = -\frac{\sigma_1 - \sigma_2}{2} 2 \sin 2\theta_p + \tau_{12} 2 \cos 2\theta_p = 0,\tag{1.31}$$

which leads to the condition

$$\tan 2\theta_p = \frac{2\tau_{12}}{\sigma_1 - \sigma_2}.\tag{1.32}$$

This equation presents two solutions θ_p and $\theta_p + \pi/2$ corresponding to two mutually orthogonal directions, called the *principal stress directions*. The maximum axial stress is found along one direction, and the minimum along the other. To define these orientations unequivocally, it is convenient to introduce the following relationships

$$\sin 2\theta_p = \frac{\tau_{12}}{\Delta}; \quad \cos 2\theta_p = \frac{\sigma_1 - \sigma_2}{\Delta},\tag{1.33}$$

where

$$\Delta = \left[\left(\frac{\sigma_1 - \sigma_2}{2} \right)^2 + \tau_{12}^2 \right]^{1/2}.\tag{1.34}$$

These relationships give a unique solution for θ_p . The maximum and minimum axial stresses, denoted σ_{1p}^* and σ_{2p}^* , respectively, act in the direction θ_p and $\theta_p + \pi/2$, respectively. These maximum and minimum axial stresses, called the *principal stresses* are found by introducing (1.33) into (1.29) to find

$$\sigma_{1p}^* = \frac{\sigma_1 + \sigma_2}{2} + \Delta; \quad \sigma_{2p}^* = \frac{\sigma_1 + \sigma_2}{2} - \Delta.\tag{1.35}$$

Note that the principal stresses are maximum and minimum values of the axial stress in an algebraic sense. It is however possible to have $|\sigma_{2p}^*| > |\sigma_{1p}^*|$. The shearing stress in the principal directions vanishes as can be verified by introducing (1.33) into (1.29).

It is also interesting to find the orientation of the faces leading to the maximum value of the shearing stress. The orientation θ_s corresponding to this condition satisfies the following relation

$$\frac{d\tau_{12}^*}{d\theta} = -\frac{\sigma_1 - \sigma_2}{2} 2 \cos 2\theta_s - \tau_{12} 2 \sin 2\theta_s = 0, \quad (1.36)$$

or

$$\tan 2\theta_s = -\frac{2\tau_{12}}{\sigma_1 - \sigma_2}. \quad (1.37)$$

Here again, this equation presents two solutions θ_s and $\theta_s + \pi/2$ corresponding to two orthogonal faces. An orientation is unequivocally defined with the help of the following notations

$$\sin 2\theta_s = -\frac{\sigma_1 - \sigma_2}{\Delta}; \quad \cos 2\theta_s = \frac{\tau_{12}}{\Delta}. \quad (1.38)$$

Comparing (1.33) and (1.38) shows that

$$\theta_s = \theta_p - \frac{\pi}{4}. \quad (1.39)$$

This means that the faces on which the maximum shearing stresses occur are inclined at a 45° angle with respect to the principal stress directions. The maximum shearing stress is found by introducing (1.38) into (1.29) to find

$$\tau_{12s}^* = \Delta = \frac{\sigma_{1p}^* - \sigma_{2p}^*}{2}. \quad (1.40)$$

The axial stresses acting on these faces are found similarly

$$\sigma_{1s}^* = \sigma_{2s}^* = \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_{1p}^* + \sigma_{2p}^*}{2}. \quad (1.41)$$

1.2.3 Special states of stress

A stress state of great practical importance is the state of *pure shear* characterized by principal stresses of equal magnitude but opposite signs, i.e. $\sigma_{2p}^* = -\sigma_{1p}^*$ as depicted in fig. 1.6. The maximum shearing stress is found on faces inclined at a 45° angle with respect to the principal stress directions, and for (1.40) and (1.41)

$$\tau_{12s}^* = \sigma_{1p}^*; \quad \sigma_{1s}^* = \sigma_{2s}^* = 0. \quad (1.42)$$

In other words, the only non vanishing stress components on faces oriented in the θ_s direction are the shearing stresses.

Another stress state of practical importance is the hydrostatic state of stress. In this case, the principal stresses are equal, i.e. $\sigma_{1p}^* = \sigma_{2p}^* = p$, where p is the hydrostatic pressure. It follows from (1.29) that the stresses acting on a face with an arbitrary orientation are

$$\sigma_1 = \sigma_2 = p; \quad \tau_{12} = 0. \quad (1.43)$$

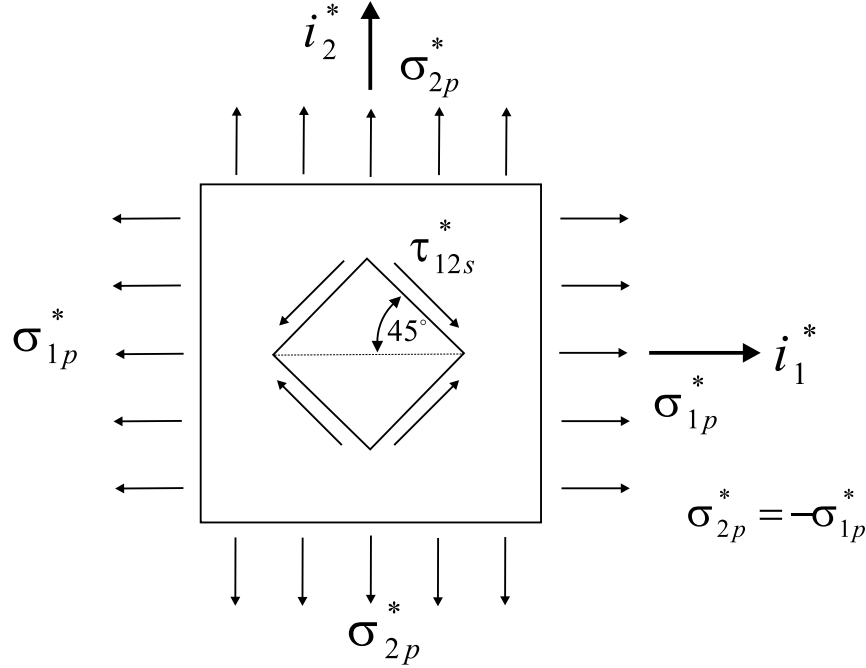


Figure 1.6: A differential element in a state of pure shear.

1.3 The concept of strain

1.3.1 The state of strain at a point

The state of strain at a point is a characterization of the deformation in the neighborhood of a material point in a solid. From a physical standpoint, straining can be viewed as a combination of stretching of material lines, and angular distortion between these material lines. As a result, two types of quantities will be used to characterize the deformation at a point: the *relative elongations* $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and the *angular distortions* γ_{23}, γ_{13} , and γ_{12} .

Consider a point P of a solid in a reference state and three material frame lines intersecting at P chosen parallel to the axes $\mathbf{i}_1, \mathbf{i}_2$, and $\mathbf{i}_3$ of a Cartesian reference frame $\mathcal{S}$, and of length dx_1, dx_2 , and dx_3 , respectively. These material lines are shown in fig. 1.7, which also shows point P in the deformed configuration of the solid. The orientation of each material line has changed, as did their length. The displacement vector from the reference to the deformed configuration is denoted $\mathbf{u}$. This vector has units of length, i.e. m , and its component in the reference frame are

$$\mathbf{u} = u_1 \mathbf{i}_1 + u_2 \mathbf{i}_2 + u_3 \mathbf{i}_3. \quad (1.44)$$

The relative elongation ε_1 of material line PQ is defined as

$$\varepsilon_1 = \frac{|\mathbf{PQ}|_{\text{def}} - |\mathbf{PQ}|_{\text{ref}}}{|\mathbf{PQ}|_{\text{ref}}}, \quad (1.45)$$

where the subscripts $(\cdot)_{\text{ref}}$ and $(\cdot)_{\text{def}}$ are used to indicate the reference and deformed configurations, respectively. The relative elongation is a unitless quantity. The length of the

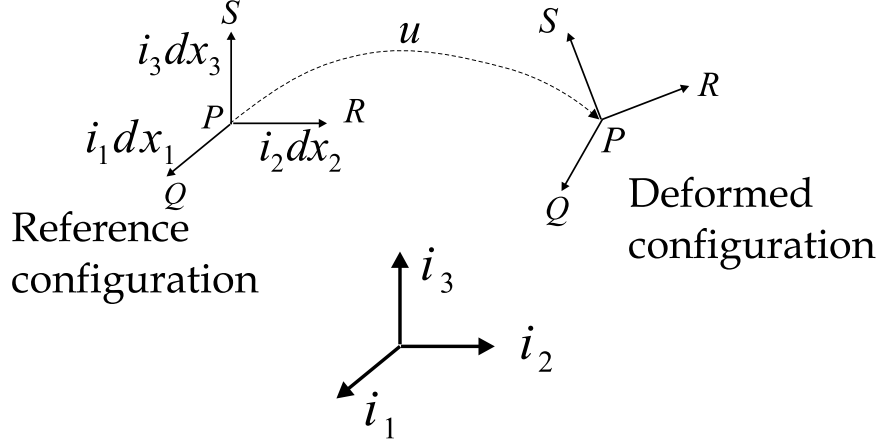


Figure 1.7: The neighbourhood of point P in the reference and deformed configurations.

material lines in the reference and deformed configuration are

$$|\mathbf{PQ}|_{\text{ref}} = |\mathbf{i}_1 dx_1| = dx_1; \quad (1.46)$$

$$|\mathbf{PQ}|_{\text{def}} = |\mathbf{i}_1 dx_1 + \frac{\partial \mathbf{u}}{\partial x_1} dx_1| = \left[1 + 2 \frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1} \right)^2 + \left(\frac{\partial u_2}{\partial x_1} \right)^2 + \left(\frac{\partial u_3}{\partial x_1} \right)^2 \right]^{1/2} dx_1. \quad (1.47)$$

The relative elongation becomes

$$\varepsilon_1 = \left[1 + 2 \frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1} \right)^2 + \left(\frac{\partial u_2}{\partial x_1} \right)^2 + \left(\frac{\partial u_3}{\partial x_1} \right)^2 \right]^{1/2} - 1. \quad (1.48)$$

A fundamental assumption of linear elasticity is that all displacement components remain very small so that second order terms can be neglected, i.e.

$$\begin{aligned} |u_1| \ll 1; \quad |u_2| \ll 1; \quad |u_3| \ll 1; \\ \left| \frac{\partial u_1}{\partial x_1} \right| \ll 1; \quad \left| \frac{\partial u_2}{\partial x_1} \right| \ll 1; \quad \left| \frac{\partial u_3}{\partial x_1} \right| \ll 1; \quad \left| \frac{\partial u_1}{\partial x_2} \right| \ll 1; \quad \text{etc.} \end{aligned} \quad (1.49)$$

The expression for the relative elongation now reduces to

$$\varepsilon_1 \approx 1 + \frac{\partial u_1}{\partial x_1} - 1 = \frac{\partial u_1}{\partial x_1}. \quad (1.50)$$

A similar reasoning applies to material lines $\mathbf{PR}$ and $\mathbf{PS}$ to yield the expressions for the various components of relative elongation

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1}; \quad \varepsilon_2 = \frac{\partial u_2}{\partial x_2}; \quad \varepsilon_3 = \frac{\partial u_3}{\partial x_3}. \quad (1.51)$$

The angular distortion γ_{23} between two material lines $\mathbf{PR}$ and $\mathbf{PS}$ is defined as the change of the initially straight angle, i.e.

$$\gamma_{23} = \langle RPS \rangle_{\text{def}} - \frac{\pi}{2} \quad (1.52)$$

As for the relative elongation, the angular distortion is a unitless quantity. The sine of the angular distortion becomes

$$\sin \gamma_{23} = \sin \left(\langle RPS \rangle_{\text{def}} - \frac{\pi}{2} \right) = \cos \langle RPS \rangle_{\text{def}}. \quad (1.53)$$

The cosine of the angle between the two material lines is computed from the following trigonometric identity applied to triangle RPS in the deformed configuration

$$|\mathbf{RS}|_{\text{def}}^2 = |\mathbf{PR}|_{\text{def}}^2 + |\mathbf{PS}|_{\text{def}}^2 - 2 \cos \langle RPS \rangle_{\text{def}} |\mathbf{PR}|_{\text{def}} |\mathbf{PS}|_{\text{def}}. \quad (1.54)$$

The angular distortion becomes

$$\gamma_{23} = \arcsin \frac{|\mathbf{PR}|_{\text{def}}^2 + |\mathbf{PS}|_{\text{def}}^2 - |\mathbf{RS}|_{\text{def}}^2}{2 |\mathbf{PR}|_{\text{def}} |\mathbf{PS}|_{\text{def}}}. \quad (1.55)$$

The length of segment $\mathbf{RS}$ is readily evaluated

$$|\mathbf{RS}|_{\text{def}}^2 = \left| \left(\mathbf{i}_3 dx_3 + \mathbf{u} + \frac{\partial \mathbf{u}}{\partial x_3} dx_3 \right) - \left(\mathbf{i}_2 dx_2 + \mathbf{u} + \frac{\partial \mathbf{u}}{\partial x_2} dx_2 \right) \right|^2$$

The lengths of segments $\mathbf{PR}$ and $\mathbf{PS}$ are evaluated in a manner similar to (1.47). The angular distortion becomes

$$\gamma_{23} = \arcsin \frac{\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_1}{\partial x_2} \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_2} \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \frac{\partial u_3}{\partial x_3}}{\left[1 + 2 \frac{\partial u_2}{\partial x_2} + \left(\frac{\partial u_1}{\partial x_2} \right)^2 + \left(\frac{\partial u_2}{\partial x_2} \right)^2 + \left(\frac{\partial u_3}{\partial x_2} \right)^2 \right]^{1/2} \left[1 + 2 \frac{\partial u_3}{\partial x_3} + \left(\frac{\partial u_1}{\partial x_3} \right)^2 + \left(\frac{\partial u_2}{\partial x_3} \right)^2 + \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right]^{1/2}} \quad (1.56)$$

With the help of the small displacement assumption (1.49), this complex expression reduces to

$$\gamma_{23} = \arcsin \frac{\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}}{1 + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}} \approx \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}. \quad (1.57)$$

A similar reasoning applies to the other material lines to yield expressions for the various angular distortions

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}; \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}; \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}. \quad (1.58)$$

The relative elongations (1.51) and angular distortions (1.58) characterize the state of deformation at a point. The relative elongations are also called *direct strains* or *axial strains*, whereas the angular distortions are also called *shearing strains*. It is important to note that the *strain-displacement relationships* (1.51) and (1.58) were obtained under the small displacement assumption (1.49). If the displacements become large, expressions (1.48) and (1.56) should be used. It is clear that the small displacement assumption implies that all strain components also remain very small

$$|\varepsilon_1| \ll 1; \quad |\varepsilon_2| \ll 1; \quad |\varepsilon_3| \ll 1; \quad |\gamma_{23}| \ll 1; \quad |\gamma_{13}| \ll 1; \quad |\gamma_{12}| \ll 1; \quad (1.59)$$

Consider the block of material defined by the three vectors $\mathbf{PQ}$, $\mathbf{PR}$, and $\mathbf{PS}$. The volume of this block is $dx_1 dx_2 dx_3$. After deformation, this volume becomes

$$V \approx (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) dx_1 dx_2 dx_3 \approx (1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) dx_1 dx_2 dx_3, \quad (1.60)$$

where the higher order strain quantities were neglected in view of (1.59). The relative change in volume is now

$$\Delta = \frac{(1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) dx_1 dx_2 dx_3 - dx_1 dx_2 dx_3}{dx_1 dx_2 dx_3} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3. \quad (1.61)$$

The quantity Δ is known as the *volumetric strain* and measures the relative change in volume of the material.

1.4 Analysis of the state of strain at a point

The state of strain at a point has been characterized in the previous section by the relative elongation of three material lines and the angular distortion between them. The three material lines were selected parallel to the axes of the Cartesian reference frame $\mathcal{S}$. It is clear that the orientation of this reference frame is entirely arbitrary. A reference frame $\mathcal{S}^*$ with unit vectors $\mathbf{i}_1^*$, $\mathbf{i}_2^*$, and $\mathbf{i}_3^*$ could have been selected, and an analysis identical to that of the previous section would have led to the definition of relative elongations

$$\varepsilon_1^* = \frac{\partial u_1^*}{\partial x_1^*}; \quad \varepsilon_2^* = \frac{\partial u_2^*}{\partial x_2^*}; \quad \varepsilon_3^* = \frac{\partial u_3^*}{\partial x_3^*}, \quad (1.62)$$

and angular distortions

$$\gamma_{23}^* = \frac{\partial u_2^*}{\partial x_3^*} + \frac{\partial u_3^*}{\partial x_2^*}; \quad \gamma_{13}^* = \frac{\partial u_1^*}{\partial x_3^*} + \frac{\partial u_3^*}{\partial x_1^*}; \quad \gamma_{12}^* = \frac{\partial u_1^*}{\partial x_2^*} + \frac{\partial u_2^*}{\partial x_1^*}. \quad (1.63)$$

Though expressed in different reference frames, the strain displacements equations (1.51) and (1.58) or (1.62) and (1.63) both characterize the state of deformation at a point of the body. Therefore, the strain components resolved in the two reference frames should be closely related.

1.4.1 Rotation of strains

We propose to relate the components of strain in two different Cartesian systems $\mathcal{S}$ and $\mathcal{S}^*$. To simplify the derivation, a two dimensional case is considered first. Fig. 1.8 depicts the two axis systems $\mathcal{S}$ and $\mathcal{S}^*$ defined in the previous sections, and an arbitrary point P. Let the coordinates of point P be (x_1, x_2) and (x_1^*, x_2^*) in the $\mathcal{S}$ and $\mathcal{S}^*$ systems, respectively. These coordinates are related as follows:

$$x_1 = x_1^* \cos \theta - x_2^* \sin \theta; \quad x_2 = x_1^* \sin \theta + x_2^* \cos \theta. \quad (1.64)$$

Consider now the strain component ε_1^* in the $\mathcal{S}^*$ system. The chain rule for derivatives yields:

$$\varepsilon_1^* = \frac{\partial u_1^*}{\partial x_1^*} = \frac{\partial u_1^*}{\partial x_1} \frac{\partial x_1}{\partial x_1^*} + \frac{\partial u_1^*}{\partial x_2} \frac{\partial x_2}{\partial x_1^*}.$$

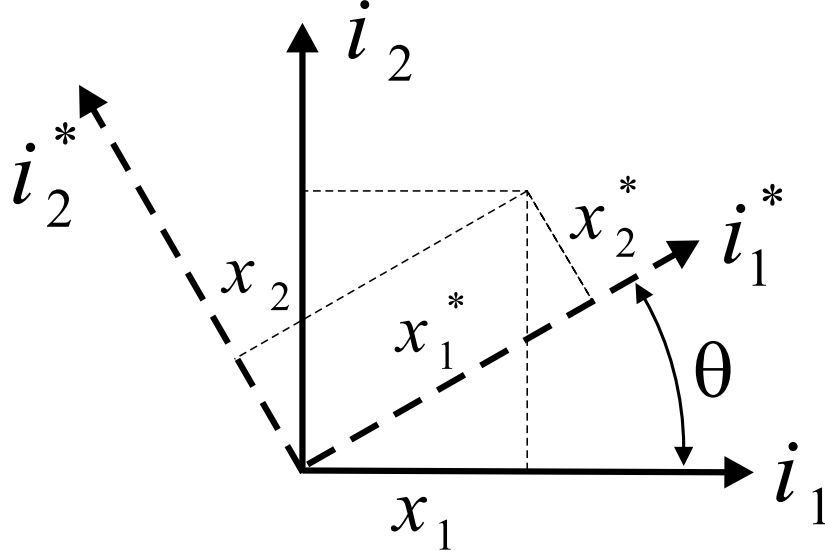


Figure 1.8: Position of a point in two coordinate systems.

Equations (1.64) show that $\partial x_1 / \partial x_1^* = \cos \theta$ and $\partial x_2 / \partial x_1^* = \sin \theta$. The axial strain component now becomes:

$$\varepsilon_1^* = \cos \theta \frac{\partial u_1^*}{\partial x_1} + \sin \theta \frac{\partial u_1^*}{\partial x_2}. \quad (1.65)$$

The displacement components of point P denoted (u_1, u_2) and (u_1^*, u_2^*) in the $\mathcal{S}$ and $\mathcal{S}^*$ systems, respectively, are related through equations similar to (1.64), namely

$$u_1^* = u_1 \cos \theta + u_2 \sin \theta; \quad u_2^* = -u_1 \sin \theta + u_2 \cos \theta. \quad (1.66)$$

Introducing these expressions into (1.65) yields the axial strain component ε_1^* in terms of the strain components in the $\mathcal{S}$ system

$$\varepsilon_1^* = \cos^2 \theta \varepsilon_1 + \sin^2 \theta \varepsilon_2 + \sin \theta \cos \theta \gamma_{12}. \quad (1.67)$$

Similar developments yield the various in-plane strain components. The results can be summarized in the following matrix equations

$$\begin{bmatrix} \varepsilon_1^* \\ \varepsilon_2^* \\ \gamma_{12}^* \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}. \quad (1.68)$$

These relations can be readily inverted to find

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1^* \\ \varepsilon_2^* \\ \gamma_{12}^* \end{bmatrix}. \quad (1.69)$$

With the help of basic trigonometric identities, the strain transformations rules (1.68) are written in an alternative form

$$\begin{aligned}\varepsilon_1^* &= \frac{\varepsilon_1 + \varepsilon_2}{2} + \frac{\varepsilon_1 - \varepsilon_2}{2} \cos 2\theta + \frac{\gamma_{12}}{2} \sin 2\theta; \\ \varepsilon_2^* &= \frac{\varepsilon_1 + \varepsilon_2}{2} - \frac{\varepsilon_1 - \varepsilon_2}{2} \cos 2\theta - \frac{\gamma_{12}}{2} \sin 2\theta; \\ \frac{\gamma_{12}^*}{2} &= -\frac{\varepsilon_1 - \varepsilon_2}{2} \sin 2\theta + \frac{\gamma_{12}}{2} \cos 2\theta.\end{aligned}\tag{1.70}$$

This important result shows that the knowledge of the strain components ε_1 , ε_2 , and γ_{12} in two orthogonal directions allows the computation of the strain components in an arbitrary orientation. In other words, the knowledge of the strain components in two orthogonal directions fully defines the state of strain at that point.

1.4.2 Principal strains

The relative elongation in an arbitrary direction θ can be computed with the help of (1.70). It is interesting to find the orientation θ_p corresponding to the maximum or minimum relative elongation. To that effect, the derivative of ε_1^* with respect to θ is set to zero

$$\frac{d\varepsilon_1^*}{d\theta} = -\frac{\varepsilon_1 - \varepsilon_2}{2} 2 \sin 2\theta_p + \frac{\gamma_{12}}{2} 2 \cos 2\theta_p = 0,\tag{1.71}$$

which leads to the condition

$$\tan 2\theta_p = \frac{\gamma_{12}}{\varepsilon_1 - \varepsilon_2}.\tag{1.72}$$

This equation presents two solutions θ_p and $\theta_p + \pi/2$ corresponding to two mutually orthogonal directions, called the *principal strain directions*. The maximum axial strain is found along one direction, and the minimum along the other. To define these orientation unequivocally it is convenient to introduce the following relationships

$$\sin 2\theta_p = \frac{\gamma_{12}/2}{\Delta}; \quad \cos 2\theta_p = \frac{\varepsilon_1 - \varepsilon_2}{\Delta},\tag{1.73}$$

where

$$\Delta = \left[\left(\frac{\varepsilon_1 - \varepsilon_2}{2} \right)^2 + \left(\frac{\gamma_{12}}{2} \right)^2 \right]^{1/2}.\tag{1.74}$$

These relationships give a unique solution for θ_p . The maximum and minimum axial strains, denoted ε_{1p}^* and ε_{2p}^* , respectively, act in the direction θ_p and $\theta_p + \pi/2$, respectively. These maximum and minimum axial strains, called the *principal strains* are found by introducing (1.73) into (1.70) to find

$$\varepsilon_{1p}^* = \frac{\varepsilon_1 + \varepsilon_2}{2} + \Delta; \quad \varepsilon_{2p}^* = \frac{\varepsilon_1 + \varepsilon_2}{2} - \Delta.\tag{1.75}$$

The shearing strain in the principal directions vanishes as can be verified by introducing (1.73) into (1.70).

1.4.3 Strain compatibility equations

The six components of strain are a function of three displacement components. Hence, the strain components are not independent and must satisfy a set of relationships called *strain compatibility equations*. Consider the following derivatives of the shear strain components

$$\frac{\partial^2 \gamma_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2}{\partial x_2 \partial x_3} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) = \frac{\partial^3 u_2}{\partial x_2 \partial x_3^2} + \frac{\partial^3 u_3}{\partial^2 x_2 \partial x_3} = \frac{\partial^2 \varepsilon_2}{\partial x_3^2} + \frac{\partial^2 \varepsilon_3}{\partial x_2^2}.$$

This clearly implies that the shear and axial strain components are not independent. Consider now a different set of derivatives

$$\begin{aligned} \frac{\partial^2 \varepsilon_1}{\partial x_2 \partial x_3} &= \frac{\partial^3 u_1}{\partial x_1 \partial x_2 \partial x_3}; & \frac{\partial \gamma_{23}}{\partial x_1} &= \frac{\partial^2 u_2}{\partial x_1 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1 \partial x_2}; \\ \frac{\partial \gamma_{13}}{\partial x_2} &= \frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1 \partial x_2}; & \frac{\partial \gamma_{12}}{\partial x_3} &= \frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3}, \end{aligned}$$

which imply

$$2 \frac{\partial^2 \varepsilon_1}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1} \left(-\frac{\partial \gamma_{23}}{\partial x_1} + \frac{\partial \gamma_{13}}{\partial x_2} + \frac{\partial \gamma_{12}}{\partial x_3} \right).$$

This is another relationship between the shear and axial strain components. Similar relationship can be obtained through cyclical permutations of the indices to yield the six strain compatibility equations

$$\begin{aligned} \frac{\partial^2 \gamma_{23}}{\partial x_2 \partial x_3} &= \frac{\partial^2 \varepsilon_2}{\partial x_3^2} + \frac{\partial^2 \varepsilon_3}{\partial x_2^2}, & 2 \frac{\partial^2 \varepsilon_1}{\partial x_2 \partial x_3} &= \frac{\partial}{\partial x_1} \left(-\frac{\partial \gamma_{23}}{\partial x_1} + \frac{\partial \gamma_{13}}{\partial x_2} + \frac{\partial \gamma_{12}}{\partial x_3} \right); \\ \frac{\partial^2 \gamma_{13}}{\partial x_1 \partial x_3} &= \frac{\partial^2 \varepsilon_1}{\partial x_3^2} + \frac{\partial^2 \varepsilon_3}{\partial x_1^2}, & 2 \frac{\partial^2 \varepsilon_2}{\partial x_1 \partial x_3} &= \frac{\partial}{\partial x_2} \left(+\frac{\partial \gamma_{23}}{\partial x_1} - \frac{\partial \gamma_{13}}{\partial x_2} + \frac{\partial \gamma_{12}}{\partial x_3} \right); \\ \frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2} &= \frac{\partial^2 \varepsilon_1}{\partial x_2^2} + \frac{\partial^2 \varepsilon_2}{\partial x_1^2}, & 2 \frac{\partial^2 \varepsilon_3}{\partial x_1 \partial x_2} &= \frac{\partial}{\partial x_3} \left(+\frac{\partial \gamma_{23}}{\partial x_1} + \frac{\partial \gamma_{13}}{\partial x_2} - \frac{\partial \gamma_{12}}{\partial x_3} \right). \end{aligned} \quad (1.76)$$

1.4.4 Measurement of strains

The goal of the theory of elasticity is to predict the state of stress at any point of an elastic body, given the applied loading. Such predictions must be validated by measuring the state of stress at specific points, then comparing these measurements with the corresponding predictions. Unfortunately, no experimental device can measure stresses directly. An indirect measurement must be made by first measuring the state of strain, then computing the corresponding state of stress using the constitutive laws for the material.

The relative elongation at the surface of a body can be measured with the help of *strain gauges*. This device consists of a very thin electric wire glued to the surface of the solid. As the solid deforms, the length of the wire is modified, thereby changing its resistivity. An accurate electrical measurement of this resistivity change yields an accurate estimate of the length change, which in turns, allows an accurate estimate of the relative elongation in the direction of the wire. Since strain quantities are often very small, strain measurements are often labeled in *micro-strains*, which indicates a relative elongation of $\mu m/m = 10^{-6} m/m$.

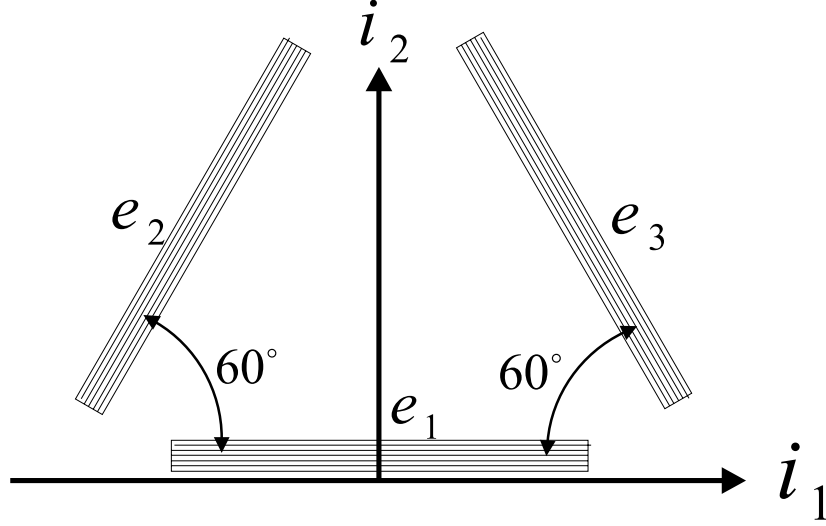


Figure 1.9: Three strain gauges at the surface of a solid.

The complete state of strain at the surface of the body can be computed from the measurement of relative elongation in three distinct directions. Fig. 1.9 shows the external surface of a body with three strain gauges forming an equilateral triangle, a common arrangement for strain gauges. Let e_1 , e_2 , and e_3 be the experimentally measured relative elongations in those three directions. With the help of (1.70) these measurements can be related to the strain components along the axes i_1 and i_2

$$\begin{aligned} e_1 &= \frac{\varepsilon_1 + \varepsilon_2}{2} + \frac{\varepsilon_1 - \varepsilon_2}{2}; \\ e_2 &= \frac{\varepsilon_1 + \varepsilon_2}{2} + \frac{\varepsilon_1 - \varepsilon_2}{2} \cos(+2 \times 60^\circ) + \frac{\gamma_{12}}{2} \sin(+2 \times 60^\circ); \\ e_3 &= \frac{\varepsilon_1 + \varepsilon_2}{2} + \frac{\varepsilon_1 - \varepsilon_2}{2} \cos(-2 \times 60^\circ) + \frac{\gamma_{12}}{2} \sin(-2 \times 60^\circ). \end{aligned} \quad (1.77)$$

The relationships can be inverted to yield the strain components in terms of the measured axial strains

$$\varepsilon_1 = e_1; \quad \varepsilon_2 = \frac{2}{3} \left(e_2 + e_3 - \frac{e_1}{2} \right); \quad \gamma_{12} = \frac{2}{\sqrt{3}} (e_2 - e_3). \quad (1.78)$$

The principal strain directions then follow from (1.73)

$$\sin 2\theta_p = \frac{e_2 - e_3}{\sqrt{3}\Delta}; \quad \cos 2\theta_p = \frac{2e_1 - e_2 - e_3}{3\Delta}, \quad (1.79)$$

and the principal strains are

$$\varepsilon_{1p}^* = \bar{e} + \Delta; \quad \varepsilon_{2p}^* = \bar{e} - \Delta, \quad (1.80)$$

where

$$\bar{e} = \frac{e_1 + e_2 + e_3}{3}; \quad \Delta = \frac{2}{3} (e_1^2 + e_2^2 + e_3^2 - e_2e_3 - e_1e_3 - e_1e_2)^{1/2}. \quad (1.81)$$

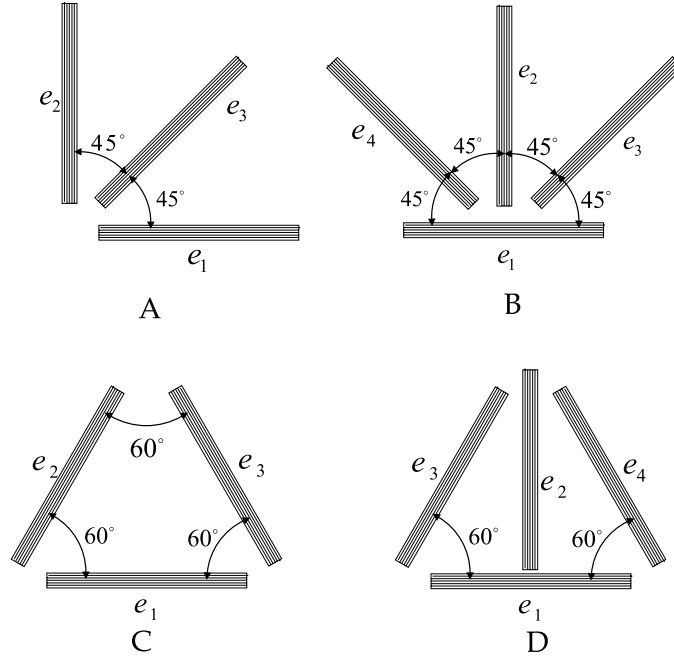


Figure 1.10: Various commonly used strain gauge arrangements.

Various commonly used strain gauge arrangements are depicted in fig. 1.10. Note that a complete evaluation of the state of strain requires the knowledge of three strain components, and thus requires three independent measurements in three distinct directions. Combinations (a) and (c) of fig. 1.10 provide three independent measurements from which the strain state can be evaluated. Combinations (b) and (d) allow four independent measurements to be made in order to provide enough information in the event of one of the gauges being out of order. If the four gauges are properly working, the redundant information is used to compensate for experimental errors.

1.5 Constitutive laws

The solution of elasticity problems requires three types of relationships. First, the equilibrium equations discussed in section (1.1.2) then the strain-displacement relationships of section (1.3.1). Finally, the stresses and strains must be related through a set of *constitutive laws*. These constitutive laws characterize the mechanical behavior of the material and consist of a set of mathematical idealization of their observed behavior.

Various types of constitutive laws exist to represent the many types of experimentally observed material behaviors. However, when the displacements and deformations of the body remain very small the stress-strain relationships are often assumed to be linear.

1.5.1 Homogeneous, isotropic, linear elastic materials

Consider the homogeneous, isotropic bar loaded by a single stress component σ_1 , as depicted in fig. 1.11. For very small applied stresses, the stress-strain relationship is assumed to be linear. This implies

$$\sigma_1 = E \varepsilon_1, \quad (1.82)$$

where the coefficient of proportionality is called *Young's modulus*. This coefficient has the same units as stress, i.e. *Pa*. This linear relationship is known as *Hooke's law*. The elongation of the bar in the direction of the applied stress is accompanied by a lateral contraction, proportional to the applied stress. The resulting deformations are

$$\varepsilon_1 = \frac{1}{E} \sigma_1; \quad \varepsilon_2 = -\frac{\nu}{E} \sigma_1; \quad \varepsilon_3 = -\frac{\nu}{E} \sigma_1, \quad (1.83)$$

where ν is called *Poisson's ratio*, a unit-less constant.

If a stress component σ_2 were applied to the same material, similar deformations would result

$$\varepsilon_1 = -\frac{\nu}{E} \sigma_2; \quad \varepsilon_2 = \frac{1}{E} \sigma_2; \quad \varepsilon_3 = -\frac{\nu}{E} \sigma_2. \quad (1.84)$$

Note that the assumption of material isotropy implies identical values of Young's modulus and Poisson's ratio in (1.83) and (1.84). Similar relationships hold for an applied stress σ_3 .

When the three stress components are simultaneously applied, the resulting deformation is the sum of the deformations obtained for each stress component applied individually, because of the assumed linear behavior of the material. Hence,

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E} [\sigma_1 - \nu(\sigma_2 + \sigma_3)]; \\ \varepsilon_2 &= \frac{1}{E} [\sigma_2 - \nu(\sigma_1 + \sigma_3)]; \\ \varepsilon_3 &= \frac{1}{E} [\sigma_3 - \nu(\sigma_1 + \sigma_2)]. \end{aligned} \quad (1.85)$$

Consider the state of pure shear described in section (1.2.3), and let the principal stresses be $\sigma_2 = -\sigma_1$, $\sigma_3 = 0$. The corresponding strain components then follow from the generalized Hooke's law (1.85)

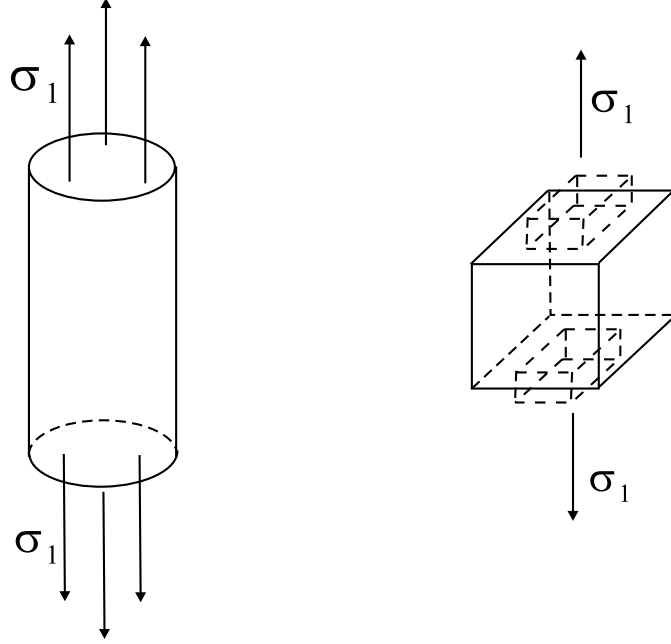
$$\varepsilon_1 = \frac{1 + \nu}{E} \sigma_1; \quad \varepsilon_2 = -\frac{1 + \nu}{E} \sigma_1; \quad \gamma_{12} = 0. \quad (1.86)$$

The shearing strain vanishes since the shearing stress is zero in the principal stress directions. In the analysis of the pure shear stress state, the state of stress on faces oriented at a 45° angle with respect to the principal stress directions was found to be

$$\tau_{12s}^* = \sigma_1; \quad \sigma_{1s}^* = \sigma_{2s}^* = 0. \quad (1.87)$$

On the other hand, the strains along the same orientations are readily obtained from (1.70), for $\theta_s = -45^\circ$

$$\gamma_{12s}^* = \frac{\varepsilon_1 - \varepsilon_2}{2} \sigma_1 = 2 \frac{1 + \nu}{2} \sigma_1; \quad \varepsilon_{1s}^* = \varepsilon_{2s}^* = 0. \quad (1.88)$$

Figure 1.11: Homogeneous bar loaded by a single stress component σ_1

The relationship between τ_{12s}^* and γ_{12s}^* is then obtained by comparing the above results to find

$$\gamma_{12s}^* = \frac{1}{G} \tau_{12s}^*, \quad (1.89)$$

where

$$G = \frac{E}{2(1 + \nu)}, \quad (1.90)$$

is called the *shearing modulus*. Of course, the above reasoning could be repeated for a state of pure shear in various planes, leading to similar results

$$\gamma_{23} = \frac{1}{G} \tau_{23}; \quad \gamma_{13} = \frac{1}{G} \tau_{13}; \quad \gamma_{12} = \frac{1}{G} \tau_{12}. \quad (1.91)$$

Here again, the shearing modulus is the same in all directions due to the assumed isotropy of the material. The shearing strain-shearing stress relationships (1.91) were established for the case of pure shear. However, they remain valid for more complex stress states involving axial stresses, because, in view of (1.85), axial stresses create no shearing strains. Similarly, the generalized Hooke's law, eq. (1.85), was established when only axial stresses are applied. They do remain valid for more complex stress states involving shearing stresses because, in view of eq. (1.91), shearing stresses create no axial strains. The constitutive laws, eqs. (1.85) and (1.91) are often called the *generalized Hooke's laws*.

In summary, a homogeneous, linear elastic, isotropic material is characterized by the constitutive laws (1.85) and (1.91). Only two material parameters are involved in these laws, Young's modulus E , and Poisson's ratio ν . The shearing modulus can be evaluated from (1.90).

The volumetric strain is readily evaluated with the help of eq. (1.61)

$$\Delta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1 - 2\nu}{E} (\sigma_1 + \sigma_2 + \sigma_3) \quad (1.92)$$

In the special case of an applied hydrostatic pressure $\sigma_1 = \sigma_2 = \sigma_3 = p$, a linear relationship is found between the applied pressure and the resulting volumetric strain

$$p = \kappa \Delta \quad (1.93)$$

where

$$\kappa = \frac{1}{3(1 - 2\nu)}, \quad (1.94)$$

is known as the *bulk modulus* of the material. When Poisson's ratio approaches a value of $1/2$, the bulk modulus approaches infinity, implying the vanishing of the volumetric strain under an applied pressure. Such a material is called an *incompressible material*. Rubbers are nearly incompressible materials and metals undergoing plastic deformations are often assumed to be nearly incompressible.

Chapter 2

Tension and Compression

2.1 Bar under constant axial force

Consider an idealized problem consisting of an infinitely long, homogeneous bar with constant properties along its span, as depicted in fig. 2.1, and subjected to end loads P . Since the axial load and physical properties are constant along the span, the deformation of the bar must be identical at all points along its span. Consider now an initially plane cross-section $\mathcal{S}$. All the material particles which form $\mathcal{S}$ before deformation form the surface $\mathcal{S}'$ after deformation. The symmetry of the problem requires the two halves of the deformed bar to be identical, resulting in the symmetry of the deformed cross-section with respect to mn . This condition can only be satisfied if the deformed cross-section is a plane that remains normal to the axis of the bar.

For the more realistic problem of a bar of finite length, the above conclusion still holds except for the portions of the bar near the end points where a complex stress distribution will occur as a result of the load application mechanism. Consider now the section nm shown on fig. 2.1. Due to the homogeneity of the bar, the deformation must be identical at all points of the section, and the axial stress σ_1 is then

$$\sigma_1 = \frac{P}{\Omega}, \quad (2.1)$$

where Ω is the cross sectional area of the bar. If δ is the elongation of the bar of length L , the axial strain ε_1 is

$$\varepsilon_1 = \frac{\delta}{l}, \quad (2.2)$$

If the applied load is small enough, deformations and stresses are proportional, as implied by Hooke's law, eq. (1.82)

$$\sigma_1 = E\varepsilon_1, \quad (2.3)$$

where E is Young's modulus. The elongation of the bar resulting from the application of the load is easily found as

$$\delta = \frac{PL}{E\Omega}. \quad (2.4)$$

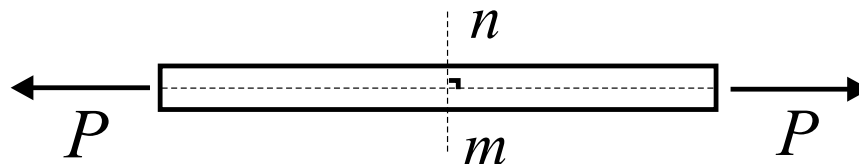


Figure 2.1: Infinitely long bar under end loads.

The above results are valid for both tensile and compressive load. However, in the case of compressive loads, the equilibrium configuration of the bar might become unstable as the load increases and the bar will buckle laterally.

2.2 Allowable stress

A central problem of structural analysis is to determine the optimal configuration of a structure subjected to specific loads. The design will be influenced by many factors associated with various structural characteristics, such as

1. *The strength of the structure.* When the local stress in the structure exceeds a specific value, the material will break or sustain permanent damage such as cracks or plastic deformations.
2. *The elastic deformations of the structure* under load. Even when subjected to small loads, a structure can present undesirable levels of elastic deformations.
3. *The dynamics characteristics of the structure.* If the structure is subjected to dynamic loads, the time history of its response becomes important. More often than not, its natural frequencies must be carefully considered so as to avoid resonance. For aerospace structures, aeroelastic phenomena such as flutter, will put stringent requirements on the natural torsional frequencies of wings and fuselages.
4. *The stability characteristics of the structure.* When parts of the structures are subjected to compressive loads, the equilibrium configuration can become unstable, resulting in buckling. During level flight, the upper skin of a wing of an aircraft is subjected to compressive loads. Avoiding buckling of this wing skin considerably affects wing design.
5. *The time dependent deformations of the structure associated with creep* of the constitutive materials. Creep considerations play an important role in aircraft engine design.

The strength of a structure is the focus of the present section, although a good design must evaluate all the above characteristics. A structure is said to fail if it breaks, collapses, or develops significant permanent damage. Clearly, the applied service loads must be less, and often much less, than those corresponding to failure. The main reason for decreasing service loads is due to the numerous uncertainties about the problem. Among these are

1. The actual magnitude of the applied service loads is not accurately known. In an aircraft, maneuver loads or those associated with a rough landing condition cannot be precisely evaluated. Accidental overloads might also take place during flight or maintenance of the aircraft.
2. The strength of materials presents statistical characteristics. Measurements of the strengths of two nominally identical samples of aluminum will be different due to material inhomogeneities, processing difference, and experimental errors,
3. Manufacturing variability also plays an important role. For instance, machining fittings of complex shapes is a delicate operation. Dimensional tolerances might vary from part to part; the strength of the resulting material might not be equal to that measured in laboratory samples, and quality control sometimes fails to detect some types of defects in the manufactured part.
4. The strength of the material might decrease in time due corrosion, wear, of the presence of a chemically aggressive environment such as fuels or solvents.
5. Finally, if failure is predicted based on the value of internal stresses in the structure, the predicted value of these stresses might be very different from their actual value. This is due to the fact that internal stresses are evaluated from the applied loads based on simplifying assumptions.

Consequently, service loads must be limited to a conservative level, and as the uncertainty about the problem increases, so must the level of conservatism in the design. It is common practice to account for all these uncertainties by defining a *load factor*

$$\text{load factor} = \frac{\text{failure load}}{\text{service load}}. \quad (2.5)$$

Of course, the load factor should be larger than unity, and is sometimes as large as 10. Engineering judgment must be carefully exercised in choosing this load factor. If a low value is selected, the likelihood of accidental failure will increase, whereas for low values, the design might be too expensive or too heavy for its intended purpose.

The load factor might be viewed as a *factor of safety* with respect to failure: limiting the service loads to a fraction of the failure loads implies a safe operation of the structure. However, using the load factor as a factor of safety is not always practical because the failure load is often unknown. Indeed, it is not practical, nor cost effective to test all structures to failure to determine the failure load. A more common approach is to compute the local stresses induced by the applied loads and limit these local stresses to an allowable level. This can be written as

$$\text{allowable stress} = \frac{\text{yield stress}}{\text{safety factor}}. \quad (2.6)$$

or

$$\sigma_{\text{allow}} = \frac{\sigma_y}{n}, \quad (2.7)$$

where σ_{allow} is the allowable stress, σ_y the yield stress of the material, and n the factor of safety. This definition is adequate for ductile material, *i.e.* materials undergoing large

strains before failure. Once the yield stress is reached, permanent deformation occur. On the other hand, brittle material, such as concrete, stone, or ceramic materials, will fail without appreciable prior deformations. For such materials, the following definition of the allowable stress is more appropriate

$$\text{allowable stress} = \frac{\text{ultimate stress}}{\text{safety factor}}, \quad (2.8)$$

where the ultimate stress is the failure stress for the material.

In summary, the stress level σ that a structure is subjected to during service should be smaller than the allowable stress, leading to the following strength criterion

$$\sigma \leq \sigma_{\text{allow}}. \quad (2.9)$$

For some materials, the allowable stress in tension and in compression are different, and the actual stress level should then be compared to the appropriate allowable stress.

2.3 Hyperstatic systems

It is often the case that the forces acting in the components of a structure cannot be determined by the sole equations of equilibrium of the system. Such systems are called *statically indeterminate* or *hyperstatic* structures. Fig. 2.2 shows a three bar truss subjected to a load P at point O where the three bars are connected together. The cross-sectional areas of the homogeneous bars are Ω_A , Ω_B , and Ω_C , for the bars attached at points A , B , and C , respectively, with $\Omega_A = \Omega_C$. Let F_A , F_B , and F_C be the forces acting in the bars attached at points A , B , and C , respectively. Expressing the horizontal equilibrium of the forces acting on a free body diagram of the entire structure yields $F_A = F_C$, as expected by the symmetry of the problem. Vertical equilibrium then yields $F_B + 2F_A \cos \Theta = P$. The two unknown forces F_B and F_A cannot be solely determined from equilibrium considerations.

If Δ_B is the elongation of the vertical bar, see fig. 2.2, the elongations in the other bars are $\Delta_A = \Delta_C = \Delta_B \cos \Theta$, assuming these elongations are much smaller than the length of the bars. The strains, and corresponding stresses are then

$$\varepsilon_B = \frac{\Delta_B}{L}, \quad \sigma_B = \frac{E_B \Delta_B}{L}, \quad (2.10)$$

for the vertical bar, and

$$\varepsilon_A = \varepsilon_C = \frac{\Delta_B \cos^2 \Theta}{L}, \quad \sigma_A = \sigma_C = \frac{E_A \Delta_B \cos^2 \Theta}{L}, \quad (2.11)$$

for the other two bars. E_B is Young's modulus for the vertical bar, and $E_A = E_C$ that for the other two. The forces in the bars now become

$$F_B = \Omega_B \sigma_B = \frac{\Omega_B E_B \Delta_B}{L}, \quad F_A = F_C = \Omega_A \sigma_A = \frac{\Omega_A E_A \Delta_B \cos^2 \Theta}{L}. \quad (2.12)$$

These relationships show that the forces in the bars remain in the following proportion

$$F_A = F_B \cos^2 \Theta \frac{E_A \Omega_A}{E_B \Omega_B}. \quad (2.13)$$

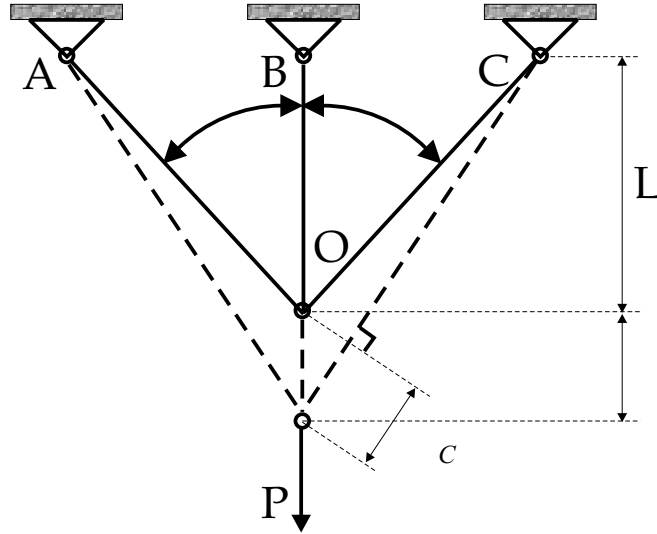


Figure 2.2: Three bar truss.

Introducing this result into the vertical equilibrium equation then yields

$$F_B = \frac{P}{1 + 2 \cos^3 \Theta \frac{E_A \Omega_A}{E_B \Omega_B}}. \quad (2.14)$$

It is interesting to note that the force in the bar now depends on the applied load P and orientation angle Θ , as expected, but also on the cross-sectional areas and bar elastic properties. If the properties of all three bars are identical the load simplifies to

$$F_B = \frac{P}{1 + 2 \cos^3 \Theta}. \quad (2.15)$$

It is often the case that hyperstatic system are more structurally efficient than isostatic systems. On the other hand, they present two important drawbacks: they are sensitive to dimensional imperfections and thermal loading. Assume that the vertical bar is too long. The only way to assemble the system would then be compress the vertical bar, assemble it, then release the system. Even when no loads are applied, the system will be subjected to *initial stresses* present in all three bars (compressive stresses in the vertical bar, and tensile stresses in the inclined bars). Initial stresses are also called *residual stresses*. If the system is unequally heated, internal stresses will also appear due to the resulting unequal length of the bars.

2.4 Thermal stresses

When an sample of material is heated its dimensions will change. Under heat, a homogeneous, isotropic material will expand equally in all directions and

$$\varepsilon^T = \alpha(\Delta T), \quad (2.16)$$

where ε^T is called the *thermal strain* and α the *coefficient of thermal expansion* which is, in general a function of ΔT , the increase in temperature. For most materials thermal expansion results from temperature increases, whereas temperature decreases result in thermal shrinkage. In that case the coefficient of thermal expansion is a positive number. There are, however, notable exceptions: the transition from water to ice under decreasing temperature is accompanied by an increase in volume. For small temperature changes, it is often adequate to assume that the coefficient of thermal expansion is a constant, *i.e.*

$$\varepsilon^T = \alpha \Delta T. \quad (2.17)$$

It is important to note that thermal strains do not generate any internal stresses, in contrast with mechanical strains that are related to internal stresses through the material constitutive laws.

Consider again the three bar truss depicted in fig. 2.2, but this time there are no applied loads. Assume now that the system was assembled when all components were at common temperature T_0 and that no initial stress was present at that point. Now, the vertical bar is heated to a temperature $T_1 > T_0$. This situation is equivalent to the vertical bar being too long by an amount $\alpha(T_1 - T_0)L$. This thermal effect give rise to mechanical strains, and the associated mechanical stresses. The elongation of the vertical bar is now

$$\Delta_B = \alpha L \Delta T + \Delta_m, \quad (2.18)$$

where Δ_m is the elongation due to the unknown mechanical forces that will appear in the system. The strains in the system are

$$\varepsilon_B = \frac{\Delta_B}{L}, \quad \varepsilon_A = \varepsilon_C = \frac{\Delta_B \cos^2 \Theta}{L}. \quad (2.19)$$

The corresponding stresses are

$$\sigma_B = \frac{E \Delta_m}{L}, \quad \sigma_A = \sigma_C = \frac{E \Delta_B \cos^2 \Theta}{L}. \quad (2.20)$$

For simplicity, the properties of all three bars are assumed to be identical. Note that the sole mechanical part of the elongation contributes to the stressing of the vertical bar. The forces in the bar are now

$$F_B = \Omega \sigma_B = \frac{\Omega E (\Delta_B - \alpha L \Delta T)}{L}, \quad F_A = F_C = \Omega \sigma_A = \frac{\Omega E \Delta_B \cos^2 \Theta}{L}. \quad (2.21)$$

The vertical equilibrium equation is now $F_B + 2F_A \cos \Theta = 0$, and introducing the forces yields

$$\Delta_B = \frac{\alpha L \Delta T}{1 + 2 \cos^3 \Theta} \quad (2.22)$$

The force in the bars becomes

$$F_B = - \frac{E \Omega \alpha \Delta T}{1 + \frac{1}{2 \cos^3 \Theta}}, \quad F_A = \frac{E \Omega \alpha \Delta T}{2 \cos \Theta (1 + \frac{1}{2 \cos^3 \Theta})}, \quad (2.23)$$

As expected, the force in the vertical bar is compressive, in contrast with the tensile loads present in the inclined bars.

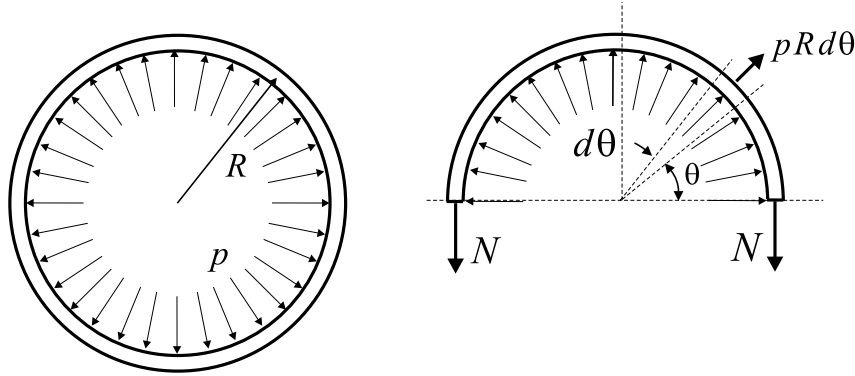


Figure 2.3: Thin ring under internal pressure.

2.5 Thin rings and tubes

Consider the thin-walled ring of radius R subjected to an internal pressure p , as depicted in fig. 2.3. This could also model a unit length section of a thin-walled tube of the same dimensions. As a result of the internal pressure, a hoop force N will act in the ring. The free body diagram of the upper part of the ring shown in fig. 2.3 yields the following equilibrium equation for the forces acting in the vertical direction

$$2N = 2 \int_0^{\pi/2} pR \sin \theta \, d\theta = 2pR, \quad (2.24)$$

or $N = pR$. For thin-walled rings, it is reasonable to assume that the hoop stresses are uniformly distributed through the thickness of the wall, *i.e.* $N = e\sigma_h$, where e is the wall thickness and σ_h the hoop stress. The hoop stress is now easily related to the applied pressure

$$\sigma_h = \frac{pR}{e}. \quad (2.25)$$

For a linear elastic material, the hoop strain ε_h is then

$$\varepsilon_h = \frac{\sigma_h}{E} = \frac{pR}{Ee} \quad (2.26)$$

and the radius of the ring increases by an amount

$$\Delta R = \frac{pR^2}{Ee}. \quad (2.27)$$

2.6 Tension or compression in two or three directions

Consider now a pressure vessel consisting of a cylindrical tube closed by spherical end caps, as depicted in fig. 2.4. The resultant F of the pressure loading on the end caps is independent of their shape and is equal to $p\pi R^2$. For a thin-walled pressure vessel one can again assume

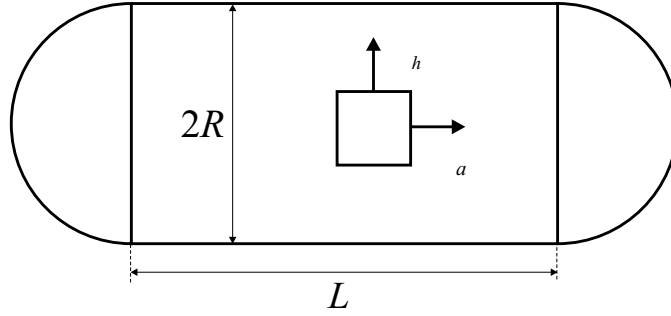


Figure 2.4: Pressure vessel under internal pressure.

that the stresses along the axis of the vessel σ_a will be uniformly distributed through the wall thickness

$$\sigma_a = \frac{p\pi R^2}{2\pi R e} = \frac{pR}{2e} = \frac{\sigma_h}{2}. \quad (2.28)$$

It should be noted that a radial stress component σ_r also exist. This stress acts along the radius of the cylindrical part of the vessel. $\sigma_r = -p$ on the internal surface of the vessel, and $\sigma_r = 0$ on the external surface. In most practical designs the ratio R/e is a very large quantity and hence $\sigma_r \ll \sigma_h = 2\sigma_a$. Consequently, the radial stress is generally ignored.

If the material can be assumed to behave as a linear elastic material, Hooke's law, eq. (1.85) implies

$$\varepsilon_h = \frac{1}{E} (\sigma_h - \nu\sigma_a) = \frac{\sigma_h}{E} (1 - \frac{\nu}{2}), \quad \varepsilon_a = \frac{1}{E} (\sigma_a - \nu\sigma_c) = \frac{\sigma_h}{E} (\frac{1}{2} - \nu). \quad (2.29)$$

Finally, the radial and longitudinal changes in vessel dimensions are

$$\Delta R = R\varepsilon_h = \frac{R\sigma_h}{E} (1 - \frac{\nu}{2}), \quad \Delta L = L\varepsilon_a = \frac{L\sigma_h}{E} (\frac{1}{2} - \nu), \quad (2.30)$$

respectively.

Another situation that gives rise to stresses in more than one direction is depicted in fig. 2.5, where a sample of elastic material, confined in an infinitely rigid cylinder, is subjected to an applied stress σ_3 . Since the cylinder cannot deform, $\varepsilon_1 = \varepsilon_2 = 0$. The first two equation of Hooke's law, eqs. (1.85), become

$$\sigma_1 = \sigma_2 = \frac{\nu}{1 - \nu} \sigma_3. \quad (2.31)$$

Introducing these results into the third equation of Hooke's law then yields

$$\varepsilon_3 = \frac{(1 + \nu)(1 - 2\nu)}{E(1 - \nu)} \sigma_3. \quad (2.32)$$

The apparent modulus of elasticity of the sample is defined as $E_a = \sigma_3/\varepsilon_3$, and

$$E_a = \varepsilon_3 = \frac{(1 - \nu)}{(1 + \nu)(1 - 2\nu)} E. \quad (2.33)$$

As Poisson's ratio approaches $1/2$, the normalized apparent modulus of elasticity E_a/E increases rapidly, as shown in fig. 2.6. For $\nu = 0.45$, $E_a/E = 3.79$, *i.e.* the apparent modulus of the sample is 3.79 times that of the material.

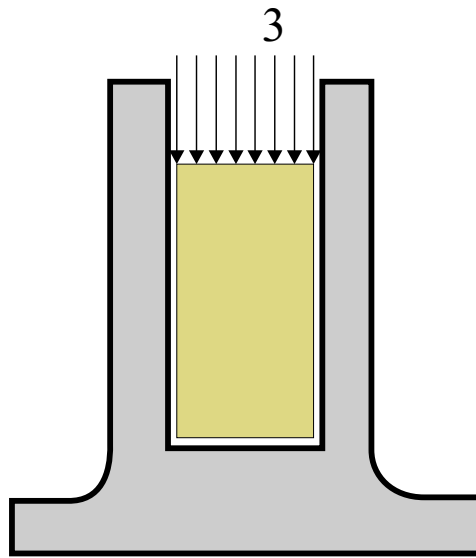


Figure 2.5: Confined sample of elastic material.

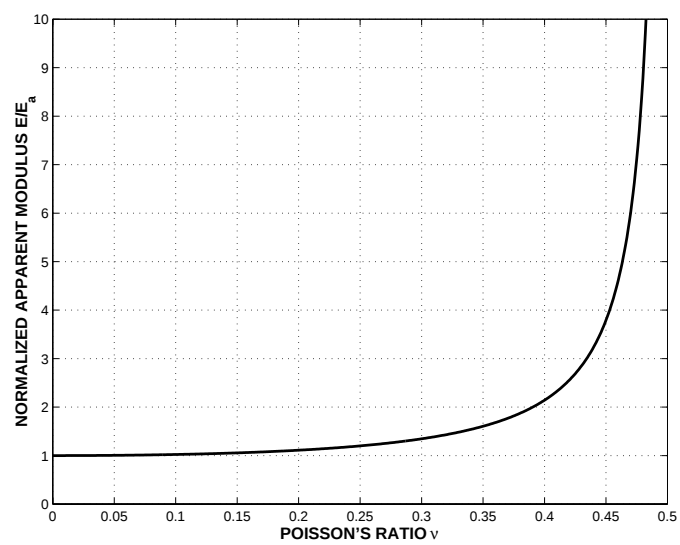


Figure 2.6: Apparent modulus of elasticity versus Poisson's ratio.

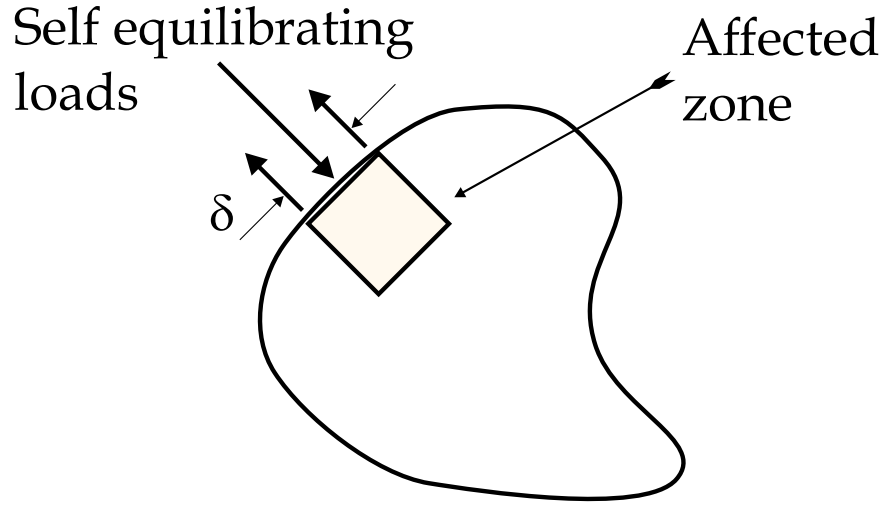


Figure 2.7: Body subjected to a set of self-equilibrating loads

2.7 Saint-Venant's principle

Consider a body subjected to a set of self-equilibrating loads, as depicted in fig. 2.7. In the vicinity of the applied loads, internal stresses will arise, as expected. However, since the net resultant of the applied load vanishes, it seems reasonable to expect their net effect to decrease away from their point of application. In other words, the effect of a set of self-equilibrating loads is expected to be localized. Typically, if the loads are applied over an area of characteristic dimension δ , the affected zone approximately extends a distance δ in all directions from the point of application.

This behavior has been observed experimentally, and is known as Saint-Venant's Principle. It can be stated as follows: *If a set of self-equilibrating loads are applied on a body over an area of characteristic dimension δ , the internal stresses resulting from these loads are only significant over a portion of the body of approximate characteristic dimension δ .* Note that this principle is rather vague, as it deals with "approximate" characteristic dimensions. It allows qualitative rather than quantitative conclusions to be drawn.

An important application of Saint-Venant's principle deals with end effects in bars and beams. In section 2.1, the stress distribution in bars subjected to end loads was studied. It is clear that the assumed uniform stress distribution over the cross-section of the bar is only valid far away from the end section of the bar. Consider fig. 2.8 where the end section of a bar of height h is subjected to a concentrated load P . This concentrated load is statically equivalent to a distributed load $p_0 = P/h$ plus a set of self-equilibrating loads, as depicted on the figure. Saint-Venant's principle implies that the self-equilibrating set of loads only affect a small zone of length h near the end of the bar.

The stress distribution in the bar, namely the uniform axial stress distribution of eq.(2.1), is identical whether the bar is subjected to end distributed or concentrated loads, except in the two end zones of length h . If the end loads are applied as a uniform distribution, the axial stresses in the bar are uniformly distributed over the cross-section at all sections. On the other hand, if the end loads are concentrated loads, the axial stresses in the bar are uniformly

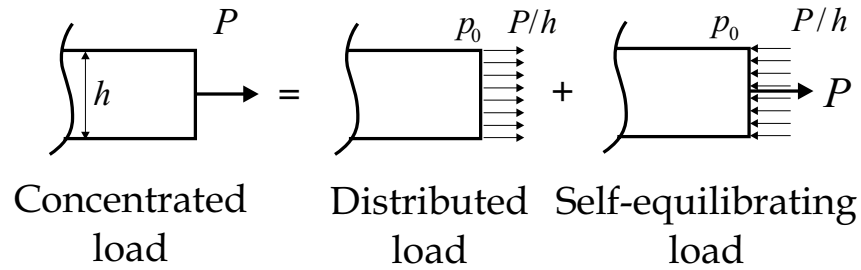


Figure 2.8: Bar subjected to an end concentrated load.

distributed over the cross-section only in the central portion of the beam. Near the end points a complex state of stress will arise; indeed, the axial stress should grow to infinity right at the point of application of the concentrated load. These end zones approximately extend a distance h at either end of the beam. The solution discussed in section 2.1 is sometimes called the *central solution*, *i.e.* the solution valid in the central portion of the bar, away from the end zones.

Chapter 3

Euler-Bernoulli Beam Theory.

Beam theory plays an important role in structural analysis as it provides the designer with a simple tool to analyze numerous structures. A beam can be defined as a structure having one of its dimensions much larger than the other two. The axis of the beam is defined along that longer dimension, and a cross-section normal to this axis is assumed to smoothly vary along the span of the beam. Civil engineering structures often consist of an assembly of structural elements such as T or I beams that can be viewed as beam grids. A large number of machine parts also are beam-like structures: lever arms, shafts, etc. Finally, several aeronautical structures such as wings and fuselages can also be treated as thin-walled box beams.

Even though more sophisticated tools, such as the finite element method, are now widely available for the stress analysis of complex structures, beam models are often used at a pre-design stage as they provide valuable insight into the behavior of the structure.

Several beam theories exist based on various assumptions and leading to various levels of accuracy. A fundamental assumption of the Euler-Bernoulli beam theory is that the cross-section of the beam is infinitely rigid in its own plane. As a result, the in-plane displacement field can be represented by two rigid body translations and one rigid body rotation.

This fundamental assumption only deals with in-plane displacements of the cross-section. Two additional assumptions deal with the out-of-plane displacements of the section: during deformation, the cross-section is assumed to remain plane and normal to the deformed axis of the beam. The implications of these assumptions are examined in the next section.

3.1 The Euler-Bernoulli Assumptions

Consider the idealized problem of an infinitely long beam with constant properties along its span depicted in fig. 3.1 subjected to end bending moments Q . The cross-section of the beam is assumed to present a plane of symmetry (the plane of the figure), and bending is taking place in that plane of symmetry.

The bending moment and physical properties are constant along the span. Hence, the deformation of the beam must be identical at all points along its axis resulting in a constant curvature. This means that the beam deforms into a circle of center O . Consider now an initially plane cross-section $\mathcal{S}$. All the material particles which form $\mathcal{S}$ before deformation form the surface $\mathcal{S}'$ after deformation. The symmetry of the problem requires the two halves

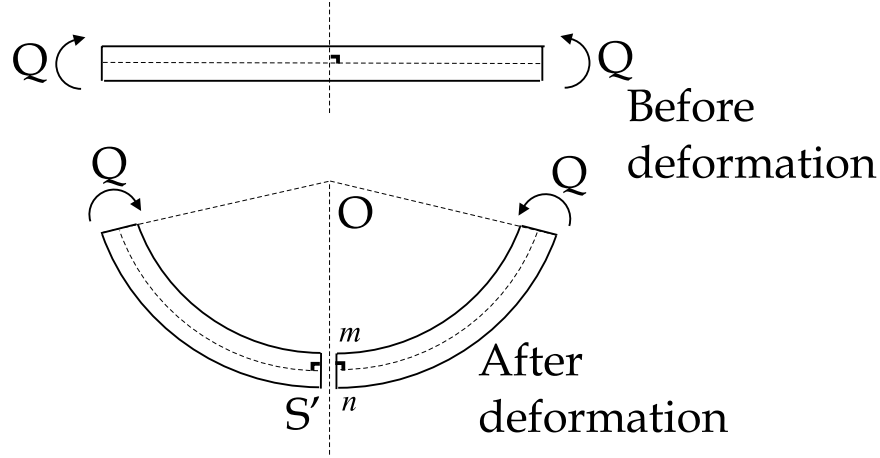


Figure 3.1: Infinitely long beam under end bending moments.

of the beam to be identical resulting in the symmetry of the deformed cross-section with respect to mn . This condition can only be satisfied if the deformed cross-section is in a plane containing the center of curvature O . This shows that the cross-section remains plane after deformation, and also remains normal to the deformed axis of the beam.

For more realistic problem, e.g. finite length beams with specific boundary conditions and applied loads, the symmetry argument that helped us in the idealized problem of an infinitely long beam is no longer applicable. However, **by analogy**, the following kinematic assumptions will be made:

Assumption 1: The cross-section is infinitely rigid in its own plane.

Assumption 2: The cross-section of a beam remains plane after deformation.

Assumption 3: The cross-section remains normal to the deformed axis of the beam.

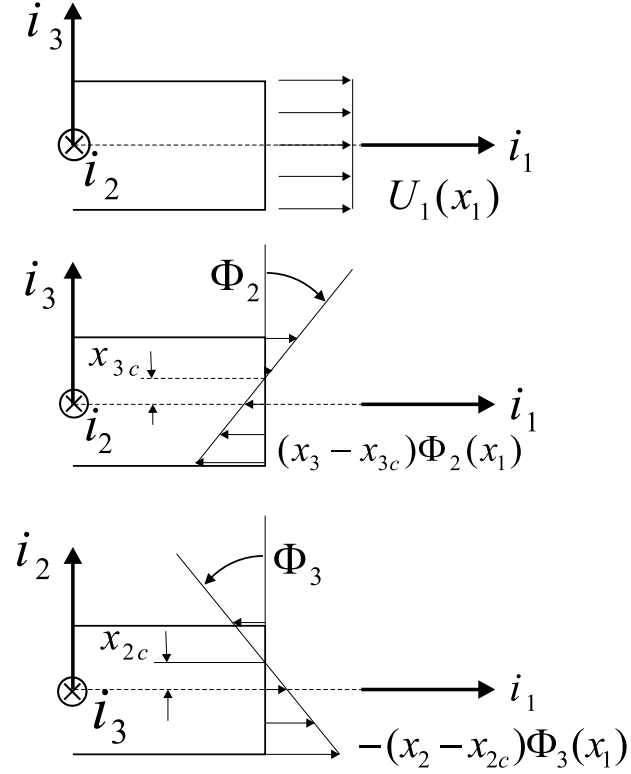
These assumptions are known as the Euler-Bernoulli assumptions for beams. Experimental measurements show that these assumptions are valid for long, slender beams made of isotropic materials with solid cross-sections. When one or more of these conditions are not met, the predictions of Euler-Bernoulli beam theory might be erroneous.

The mathematical and physical implications of the Euler-Bernoulli assumptions will now be discussed in detail.

3.2 Implications of the Euler-Bernoulli assumptions

Consider a set of unit vectors $\mathbf{i}_1, \mathbf{i}_2$, and $\mathbf{i}_3$ with coordinates x_1, x_2 , and x_3 . This set of axes is attached at a point of the beam cross-section, $\mathbf{i}_1$ is along the axis of the beam, and $\mathbf{i}_2$ and $\mathbf{i}_3$ define the plane of the cross-section. Let $u_1(x_1, x_2, x_3)$, $u_2(x_1, x_2, x_3)$, and $u_3(x_1, x_2, x_3)$ be the displacement of an arbitrary point of the beam in the $\mathbf{i}_1, \mathbf{i}_2$, and $\mathbf{i}_3$ directions, respectively.

The first Euler-Bernoulli assumption states that the cross-section is indeformable in its own plane. Hence, the displacement field in the plane of the cross-section solely consists of two rigid body translations $U_2(x_1)$ and $U_3(x_1)$



$$u_1(x_1, x_2, x_3) = U(x_1) + (x_3 - x_{3c})\Phi_2(x_1) - (x_2 - x_{2c})\Phi_3(x_1)$$

Figure 3.2: Decomposition of the axial displacement field.

$$u_2(x_1, x_2, x_3) = U_2(x_1); \quad u_3(x_1, x_2, x_3) = U_3(x_1); \quad (3.1)$$

The second Euler-Bernoulli assumption states that the cross-section remains plane after deformation. This implies an axial displacement field consisting of a rigid body translation $U_1(x_1)$, and two rigid body rotations $\Phi_2(x_1)$ and $\Phi_3(x_1)$, as depicted in fig. 3.2

$$u_1(x_1, x_2, x_3) = U_1(x_1) + (x_3 - x_{3c})\Phi_2(x_1) - (x_2 - x_{2c})\Phi_3(x_1); \quad (3.2)$$

where x_{2c} and x_{3c} are the coordinates of the centroid of the section, a point as yet undetermined. Note the sign convention: the rigid body translations of the cross-section $U_1(x_1)$, $U_2(x_1)$, and $U_3(x_1)$ are positive in the direction of the axes $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, respectively; the rigid body rotations of the cross-section $\Phi_2(x_1)$ and $\Phi_3(x_1)$ are positive about the axes $\mathbf{i}_2$ and $\mathbf{i}_3$, respectively. Fig. 3.3 depicts these various sign conventions.

The third Euler-Bernoulli assumption states the the cross-section remains normal to the deformed axis of the beam. This implies the equality of the slope of the beam and of the rotation of the section, as depicted in fig. 3.4

$$\Phi_3 = U_2'; \quad \Phi_2 = -U_3' \quad (3.3)$$

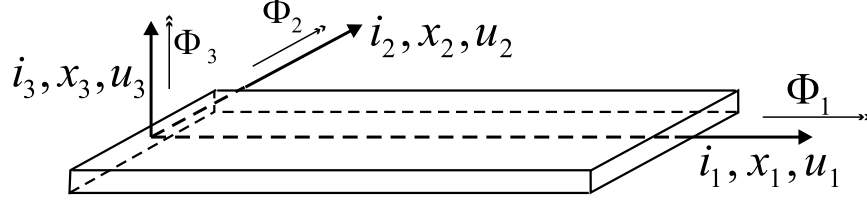


Figure 3.3: Sign convention for the displacements and rotations of a beam.

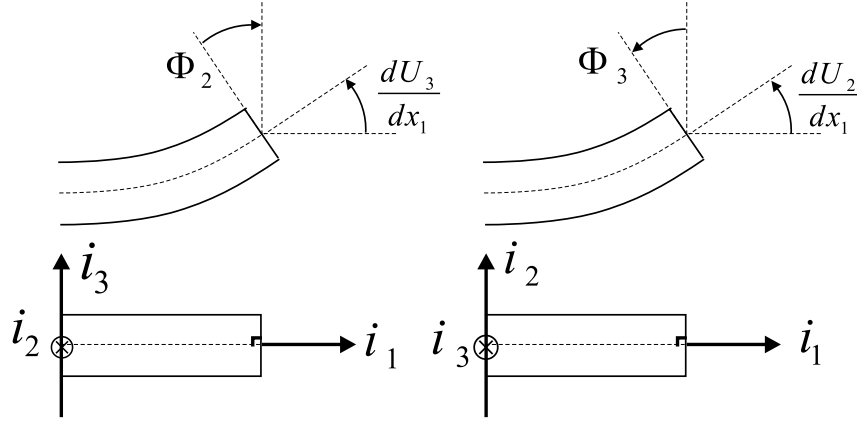


Figure 3.4: Beam slope and cross-sectional rotation.

where the notation $(.)'$ expresses a derivative with respect to x_1 . The minus sign in the second equation is a consequence of the sign convention on sectional displacements and rotations. Eqs. (3.3) can be used to eliminate the sectional rotation from the axial displacement field. The complete displacement field for Euler-Bernoulli beams writes

$$\begin{aligned} u_1(x_1, x_2, x_3) &= U_1(x_1) - (x_3 - x_{3c})U_3'(x_1) - (x_2 - x_{2c})U_2'(x_1); \\ u_2(x_1, x_2, x_3) &= U_2(x_1); \\ u_3(x_1, x_2, x_3) &= U_3(x_1). \end{aligned} \quad (3.4)$$

Clearly, the complete three-dimensional displacement field of the beam can be expressed in terms of three sectional displacements $U_1(x_1)$, $U_2(x_1)$, $U_3(x_1)$ and their span-wise derivative. This important simplification resulting from the basic Euler-Bernoulli assumptions allows the development of a one-dimensional beam theory.

The strain field can be evaluated from the displacement field defined by eqs. (3.4)

$$\varepsilon_2 = \frac{\partial u_2}{\partial x_2} = 0; \quad \varepsilon_3 = \frac{\partial u_3}{\partial x_3} = 0; \quad \gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0; \quad (3.5)$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0; \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0; \quad (3.6)$$

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1} = U_1'(x_1) - (x_3 - x_{3c})U_3''(x_1) - (x_2 - x_{2c})U_2''(x_1).$$

At this point it is convenient to introduce the following notation for the sectional strains:

$$\epsilon_1(x_1) = U_1'(x_1); \quad \kappa_2(x_1) = -U_3''(x_1); \quad \kappa_3(x_1) = U_2''(x_1). \quad (3.7)$$

where ϵ_1 is the sectional axial strain, and κ_2 and κ_3 the sectional curvature about the axes $\mathbf{i}_2$ and $\mathbf{i}_3$, respectively. With the help of these sectional strains, the axial strain distribution writes:

$$\varepsilon_1(x_1, x_2, x_3) = \epsilon_1(x_1) + (x_3 - x_{3c})\kappa_2(x_1) - (x_2 - x_{2c})\kappa_3(x_1); \quad (3.8)$$

The vanishing of the in-plane strain field is implied by eq. (3.5), and is a direct consequence of assuming the cross-section to be infinitely rigid in its own plane; the vanishing of the transverse shearing strain field is implied by eq. 3.6, and is a direct consequence of assuming the cross-section to remain normal to the deformed axis of the beam; and finally, a linear distribution of axial strains over the cross-section is implied by eq. 3.8, a direct consequence of assuming the cross-section to remain plane. Clearly, assuming a strain field of the form (3.5), (3.6), and (3.8) is entirely equivalent to the Euler-Bernoulli assumptions.

3.3 Stress resultants

The goal of a beam theory is to obtain a model of the three-dimensional beam structure which only involves sectional quantities, i.e. quantities solely dependent on the span-wise variable x_1 .

In the above section, the Euler-Bernoulli assumptions were shown to allow the description of the complete three-dimensional displacement field (3.4) of the beam in terms of three sectional displacements $U_1(x_1)$, $U_2(x_1)$, and $U_3(x_1)$, and their span-wise derivatives. Similarly, the complete three-dimensional strain field, eqs. (3.5), (3.6), and (3.8) is expressed in terms of sectional strains.

In this section, the three-dimensional stress field of the beam will be described in terms of sectional stresses called stress resultants. Three forces are defined: the axial force $F_1(x_1)$ along the $\mathbf{i}_1$ axis of the beam, and the transverse shearing forces $F_2(x_1)$ and $F_3(x_1)$ along the $\mathbf{i}_2$ and $\mathbf{i}_3$ axes, respectively. They are defined as follows

$$F_1(x_1) = \int_{\Omega} \sigma_1(x_1, x_2, x_3) d\Omega. \quad (3.9)$$

$$F_2(x_1) = \int_{\Omega} \tau_{12}(x_1, x_2, x_3) d\Omega; \quad F_3(x_1) = \int_{\Omega} \tau_{13}(x_1, x_2, x_3) d\Omega, \quad (3.10)$$

where Ω is the cross-sectional area of the beam. Next, two moments are defined: the bending moments $M_2(x_1)$ and $M_3(x_1)$ along the $\mathbf{i}_2$ and $\mathbf{i}_3$ axes, respectively. They are defined as follows

$$M_2(x_1) = \int_{\Omega} x_3 \sigma_1(x_1, x_2, x_3) d\Omega; \quad M_3(x_1) = - \int_{\Omega} x_2 \sigma_1(x_1, x_2, x_3) d\Omega. \quad (3.11)$$

Note the minus sign in the definition of $M_3(x_1)$ which is necessary to give a positive bending moment about the $\mathbf{i}_3$ axis.

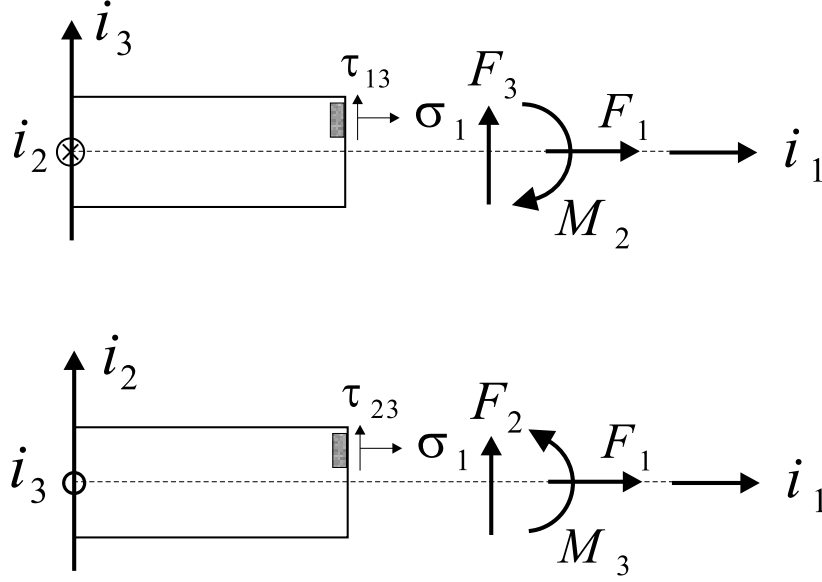


Figure 3.5: Sign convention for the sectional stress resultants.

In the above definitions, the bending moments were computed with respect to the origin of the axes. However, it will be advantageous in some cases to compute the bending moments about an axes parallel to $\mathbf{i}_2$ and $\mathbf{i}_3$ passing through a specific point of the cross-section. Specifically we define the bending moments about the centroid of the section as

$$M_2^c(x_1) = \int_{\Omega} (x_3 - x_{3c}) \sigma_1(x_1, x_2, x_3) d\Omega; \quad M_3^c(x_1) = - \int_{\Omega} (x_2 - x_{2c}) \sigma_1(x_1, x_2, x_3) d\Omega. \quad (3.12)$$

The sign convention for the forces and moments is depicted in fig. 3.5.

3.4 Beams subjected to axial loads

Consider a beam subjected to distributed axial loads p_1 as well as a concentrated axial load P_1 applied at the tip of the beam, for instance, as depicted in fig. 3.6. The units of the distributed axial load are N/m , whereas the units of the concentrated axial load are N . Under the effect of these loads the beam will extend, creating an axial displacement field $U_1(x_1)$. Furthermore, axial forces and axial stresses will be generated in the beam. This section focuses on the determination of these various quantities corresponding to a given loading applied on the beam.

3.4.1 Kinematic description

The three Euler-Bernoulli assumptions described above are made, and furthermore, it seems reasonable to assume that axial loads only cause axial displacement of the section. In other

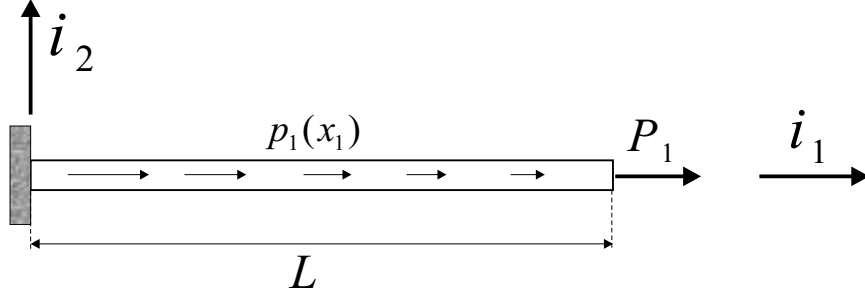


Figure 3.6: Beam subjected to axial loads.

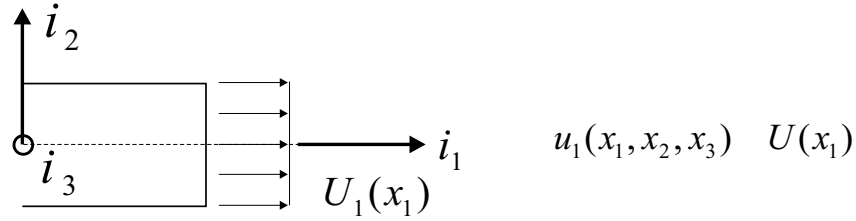


Figure 3.7: Axial displacement distribution.

words, the displacement field eq. (3.4) reduces to

$$u_1(x_1, x_2, x_3) = U_1(x_1); \quad u_2(x_1, x_2, x_3) = 0; \quad u_3(x_1, x_2, x_3) = 0, \quad (3.13)$$

and the corresponding axial strain field is

$$\varepsilon_1(x_1, x_2, x_3) = \varepsilon_1(x_1). \quad (3.14)$$

These very simple results are illustrated in fig. 3.7.

3.4.2 Sectional constitutive law

At this point, beams made of a linear elastic, isotropic material are considered. Hooke's law (1.85) describes the constitutive relationships for this type of materials. Since the cross-section does not deform in its own plane, the stresses σ_2 and σ_{33} acting in the plane of the section are far smaller than the axial stresses σ_1 and Hooke's law reduces to

$$\sigma_1(x_1, x_2, x_3) = E \varepsilon_1(x_1, x_2, x_3). \quad (3.15)$$

Introducing the axial strain distribution eq. (3.14) yields the axial stress distribution over the cross-section

$$\sigma_1(x_1, x_2, x_3) = E \varepsilon_1(x_1). \quad (3.16)$$

The axial force in the beam can be obtained by introducing this axial stress distribution into (3.9) to find

$$F_1(x_1) = \left[\int_{\Omega} E \, d\Omega \right] \varepsilon_1(x_1) = S \varepsilon_1(x_1). \quad (3.17)$$

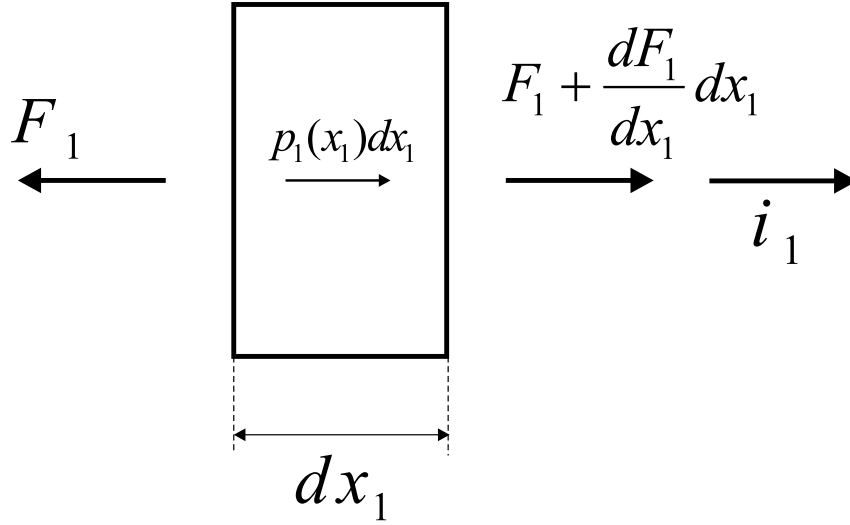


Figure 3.8: Axial forces acting on an infinitesimal slice of the beam.

Since the sectional axial strain $\epsilon_1(x_1)$ only varies along the span of the beam, it was factored out of the integral over the section. The *axial stiffness* S of the beam was defined as

$$S = \int_{\Omega} E \, d\Omega. \quad (3.18)$$

Relationship (3.17) is the constitutive law for the axial behavior of the beam. It expresses the proportionality between the axial force and the sectional axial strain, with a constant of proportionality called the axial stiffness. This constitutive law is written at the sectional level, whereas Hooke's law eq. (3.15) is written at the local, infinitesimal level.

3.4.3 Equilibrium equations

To complete the derivation, equilibrium equations must be derived for this problem. An infinitesimal slice of the beam of length dx_1 is depicted in fig. 3.8. A free body diagram of this slice yields the following equation of equilibrium

$$\frac{dF_1}{dx_1} = -p_1. \quad (3.19)$$

3.4.4 Governing equations

Finally, the governing equation of the problem is found by introducing the axial force (3.17) into the equilibrium eq. (3.19) and recalling the definition of the sectional axial strain eq. (3.7)

$$\frac{d}{dx_1} \left[S \frac{dU_1}{dx_1} \right] = -p_1. \quad (3.20)$$

This second order differential equation can be solved for the axial displacement field $U_1(x_1)$ given the axial load distribution $p_1(x_1)$.

Two boundary conditions are required for the solution of eq. (3.20), one at each end of the beam. Typical boundary conditions are:

1. A fixed (or clamped) end allows no axial displacement, i.e.:

$$U_1 = 0; \quad (3.21)$$

2. A free (unloaded) end corresponds to $F_1 = 0$, which in view of eq. (3.17) writes

$$\frac{dU_1}{dx_1} = 0; \quad (3.22)$$

3. Finally, if the end of the beam is subjected to a concentrated load P_1 , the boundary condition is $F_1 = P_1$, which becomes:

$$S \frac{dU_1}{dx_1} = P_1. \quad (3.23)$$

3.4.5 The sectional axial stiffness

The axial stiffness S of the section characterizes the stiffness of the beam when subjected to axial loads. If the beam is made of a homogeneous material, Young's modulus is identical at all points on the section and can be factored out of integral (3.18), to yield

$$S = E \Omega \quad (3.24)$$

On the other hand, if the section is made of several different materials, the axial stiffness must be computed according to eq. (3.18). An important case is that of a rectangular section of width b made of layered materials of different stiffnesses, as depicted in fig. 3.9. Assuming the material to be homogeneous within each layer with a Young modulus $E^{[i]}$ in layer i , the axial stiffness becomes

$$S = \sum_{i=1}^n E^{[i]} \int_{\Omega^{[i]}} d\Omega^{[i]} = \sum_{i=1}^n E^{[i]} b (x_2^{[i+1]} - x_2^{[i]}) \quad (3.25)$$

where n is the number of layers, $\Omega^{[i]}$ is the cross-sectional area of layer i , and $x_2^{[i]}$ and $x_2^{[i+1]}$ the coordinates of the bottom, and top planes, respectively, defining layer i . This expression clearly shows that the axial stiffness is a *weighted average* of the Young's modulus of the various layers. The weighting factor $x_2^{[i+1]} - x_2^{[i]}$ is the thickness of the layer.

3.4.6 The axial stress distribution

The determination of the local axial stress σ_1 for a given axial load p_1 is of primary interest to the designer. This can be readily obtained by eliminating the axial strain from eqs. (3.16) and (3.17) to find

$$\sigma_1(x_1, x_2, x_3) = \frac{E}{S} F_1(x_1) \quad (3.26)$$

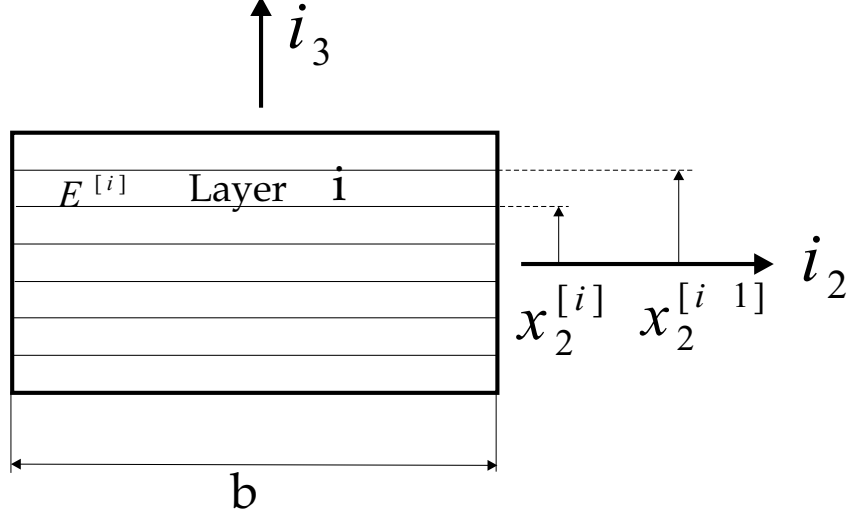


Figure 3.9: Cross-section of a beam with various layered materials.

If the beam is made of a homogeneous material, the axial stiffness is given by eq. (3.24), and eq. (3.26) then reduces to

$$\sigma_1(x_1, x_2, x_3) = \frac{F_1(x_1)}{\Omega}. \quad (3.27)$$

The axial stress is uniformly distributed over the section, and its value is independent of Young's modulus.

In contrast, the axial stress distribution for sections with various layers will vary from layer to layer. Indeed, (3.26) becomes

$$\sigma_1^{[i]}(x_1, x_2, x_3) = E^{[i]} \frac{F_1(x_1)}{S} \quad (3.28)$$

where $\sigma_1^{[i]}$ indicates the axial stress in layer i . This relationship implies that the axial stress in layer i is proportional to the modulus of that layer. Note that according to eq. (3.14) the axial strain distribution is uniform over the section, i.e. each layer is equally strained. However, the layer with a stiffer material will develop higher axial stresses. The axial stress distribution for homogeneous and layered sections are depicted in fig. 3.10.

Once the local axial stress has been determined, a strength criterion is applied to determine whether the structure can sustain the applied loads. Introducing eq. (3.26) into the strength criterion yields $E/S |F_1(x_1)| \leq \sigma_{\text{allow}}^{\text{tens}}$ or $\sigma_{\text{allow}}^{\text{comp}}$. Since the axial force varies along the span of the beam, this condition must be checked at all point along the span. In practice, it is convenient to first determine the maximum tensile and compressive axial force, denoted $F_{1 \text{ max}}^{\text{tens}}$ and $F_{1 \text{ max}}^{\text{comp}}$, respectively, then apply the strength criterion

$$\frac{E}{S} |F_{1 \text{ max}}^{\text{tens}}| \leq \sigma_{\text{allow}}^{\text{tens}}, \quad \frac{E}{S} |F_{1 \text{ max}}^{\text{comp}}| \leq \sigma_{\text{allow}}^{\text{comp}}. \quad (3.29)$$

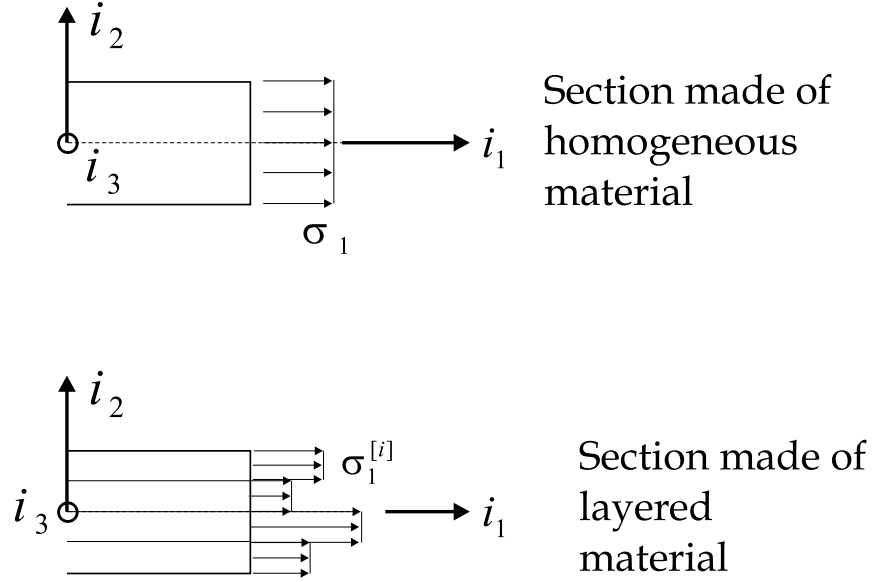


Figure 3.10: Axial stress distribution for sections made of homogeneous and layered materials.

If the axial force is compressive, buckling of the beam becomes another possible failure mode. The maximum compressive load that a beam can sustain before lateral buckling occurs is discussed in another chapter.

If the section consists of layers made of various materials, the strength of each layer will, in general, be different, and the strength criterion becomes

$$\frac{E^{[i]}}{S} |F_{1 \max}^{\text{tens}}| \leq \sigma_{\text{allow}}^{\text{tens}[i]}, \quad \frac{E^{[i]}}{S} |F_{1 \max}^{\text{comp}}| \leq \sigma_{\text{allow}}^{\text{comp}[i]}, \quad (3.30)$$

where $\sigma_{\text{allow}}^{\text{tens}[i]}$ and $\sigma_{\text{allow}}^{\text{comp}[i]}$ are the allowable stresses for layer i in tension and compression, respectively. The strength criterion must be checked for each material layer.

3.4.7 The strain energy

Consider an infinitesimal slice of the beam acted upon by an axial force F_1 . Under the effect of this force, the end cross-sections of the slice displace a differential amount dU_1 . If this deformation is proportional to the applied force, the work dW done by the axial force as it increases from zero to its present value F_1 is

$$dW = \frac{1}{2} F_1 dU_1 = \frac{1}{2} F_1 \frac{dU_1}{dx_1} dx_1.$$

This work is given by the area under the force-axial strain curve depicted in fig. 3.11. Introducing the sectional constitutive law, eq. (3.17), and the strain displacement relationship, eq. (3.7), leads to

$$dW = \frac{1}{2} S \epsilon_1^2 dx_1. \quad (3.31)$$

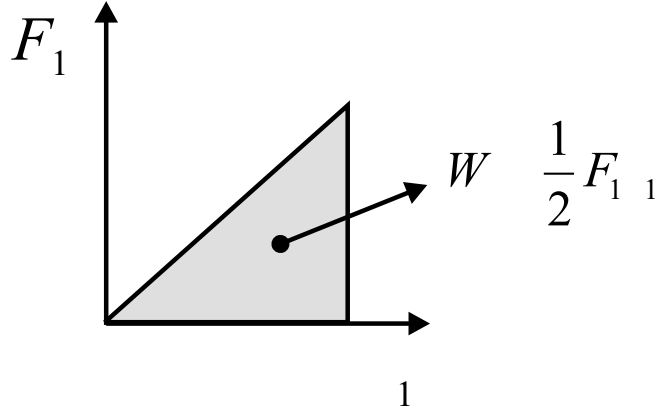


Figure 3.11: The axial force-axial strain curve for a linear elastic slice of the beam.

The work done by the force is stored in the slice of the beam in the form of elastic deformation energy, or *strain energy*. The quantity

$$a(\epsilon_1) = \frac{1}{2} S \epsilon_1^2, \quad (3.32)$$

is known as the *strain energy density function*. Eq. (3.31) expresses the fact that the work done by the axial force equals the amount of strain energy stored in the slice of the beam. The total work done by the axial force distribution in the beam is now

$$W = \frac{1}{2} \int_0^L S \epsilon_1^2 dx_1 = A(\epsilon_1), \quad (3.33)$$

and equals the total amount of strain energy stored in the beam, $A(\epsilon_1)$.

Sometimes, it is preferable to express the deformation energy stored in the beam in terms of the axial force by using eq. (3.7), to find

$$A(\epsilon_1) = \int_0^L \frac{F_1^2}{2S} dx_1 = B(F_1). \quad (3.34)$$

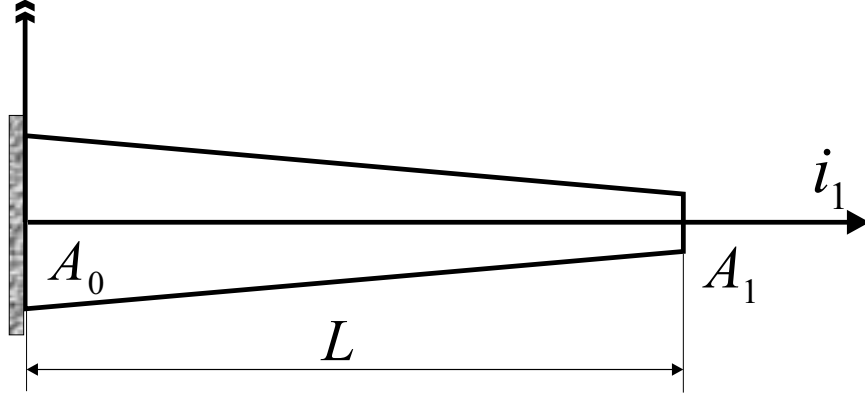
$b(F_1) = F_1^2/2S$ is known as the *stress energy density function*. $B(F_1)$ is the total deformation energy stored in the beam expressed in terms of the axial force, also called *complementary energy*.

3.4.8 Examples

Example 1: Beam under a uniform axial load

Consider the uniform, clamped beam of length L depicted in fig. 3.6 subjected to a uniform axial loading $p_1(x_1) = p_0$. The governing differential equation is given by eq. (3.20), and for the particular case at hand, writes

$$S \frac{d^2 U_1}{dx_1^2} = -p_0;$$

Figure 3.12: A helicopter blade rotating at an angular speed Ω .

$$@x_1 = 0, \quad U_1 = 0; \quad @x_1 = L, \quad S \frac{dU_1}{dx_1} = 0.$$

The solution of this differential equation is

$$U_1 = \frac{p_0 L^2}{S} \left[\left(\frac{x_1}{L} \right) - \frac{1}{2} \left(\frac{x_1}{L} \right)^2 \right]. \quad (3.35)$$

Example 2: Tapered beam under centrifugal load

A helicopter blade of length L is rotating at an angular velocity Ω about the $\mathbf{i}_2$ axis, as depicted in fig. 3.12. The blade is homogeneous and its cross-section linearly tapers from an area A_0 at the root to A_1 at the tip, *i.e.*

$$A(x_1) = A_0 + (A_1 - A_0) \frac{x_1}{L} = A_0 \left(1 - \frac{x_1}{2L} \right).$$

Consequently, the axial stiffness varies along the beam span, $S(x_1) = EA(x_1)$, where E is Young's modulus. Due to the centrifugal loading associated with the rotation, the blade is subjected to a distributed load

$$p_1(x_1) = \rho A(x_1) \Omega^2 x_1,$$

where ρ is the material density. The governing differential equation for this problem becomes

$$\frac{d}{dx_1} \left[EA_0 \left(1 - \frac{x_1}{2L} \right) \frac{dU_1}{dx_1} \right] = -\rho A_0 \left(1 - \frac{x_1}{2L} \right) \Omega^2 x_1;$$

$$@x_1 = 0, \quad U_1 = 0; \quad @x_1 = L, \quad S \frac{dU_1}{dx_1} = 0.$$

It is convenient to use the nondimensional span variable $\eta = x_1/L$ to write these equations in a more compact form

$$[(1 - \eta/2) U_1']' = -\frac{\rho \Omega^2 L^3}{E} (\eta - \eta^2/2);$$

$$@\eta = 0, \quad U_1 = 0; \quad @\eta = 1, \quad U_1' = 0,$$

where the notation $()'$ denotes a derivative with respect to η . This differential equation can be integrated once, and with the help of the boundary condition at the tip of the blade becomes

$$U_1' = \frac{\rho\Omega^2 L^3}{E} \frac{\frac{1}{3} - \frac{\eta^2}{2} + \frac{\eta^3}{6}}{1 - \eta/2} = \frac{\rho\Omega^2 L^3}{E} \left[2 + \eta - \eta^2 - \frac{1}{1 - \eta/2} \right].$$

A second integration then yields

$$U_1 = \frac{\rho\Omega^2 L^3}{3E} \left[2\eta + \frac{\eta^2}{2} - \frac{\eta^3}{3} + 2 \ln(1 - \frac{\eta}{2}) \right], \quad (3.36)$$

where the boundary condition at the root of the blade was used to evaluate the integration constant. Finally, the axial force in the blade is readily obtained from eq. (3.17)

$$F_1 = \rho A_0 \Omega^2 L^2 \left[\frac{1}{3} - \frac{\eta^2}{2} + \frac{\eta^3}{6} \right]. \quad (3.37)$$

Note the appearance of a transcendental function, the logarithm function, in the axial displacement expression. This is due to span-wise variation in axial stiffness. In practical applications, structures are subjected to complex loading conditions, and the structural properties vary dramatically along the span. Consequently, the integration of the governing differential equations becomes increasingly difficult, if not impossible.

3.4.9 Problems

Problem 3.1

Fig. 3.13 depicts an aluminum rectangular box beam of height $h = 0.30 \text{ m}$, width $b = 0.15 \text{ m}$, flange thickness $t_a = 12 \times 10^{-3} \text{ m}$, and web thickness $t_w = 5 \times 10^{-3} \text{ m}$. The beam is reinforced by two layers of unidirectional composite material of thickness $t_c = 4 \times 10^{-3} \text{ m}$. The section is subjected to an axial load $F_1 = 600 \times 10^3 \text{ N}$. The Young's moduli for the aluminum and unidirectional composite are $E_a = 73 \text{ GPa}$ and $E_c = 140 \text{ GPa}$, respectively.

1. Find the distribution of axial stress over the cross-section and sketch it.
2. Find the magnitude and location of the maximum axial stress in the aluminum and composite layer.
3. Assume the applied loads grow in a proportional manner, *i.e.* the applied loads are λF_1 , λM_2^c , and λM_3^c . If the allowable stress for the aluminum and unidirectional composite are $\sigma_a^{\text{allow}} = 400 \text{ MPa}$ and $\sigma_c^{\text{allow}} = 1500 \text{ MPa}$, respectively, find the maximum loading factor λ_{Max} .
4. Sketch the distribution of axial strain over the cross-section.

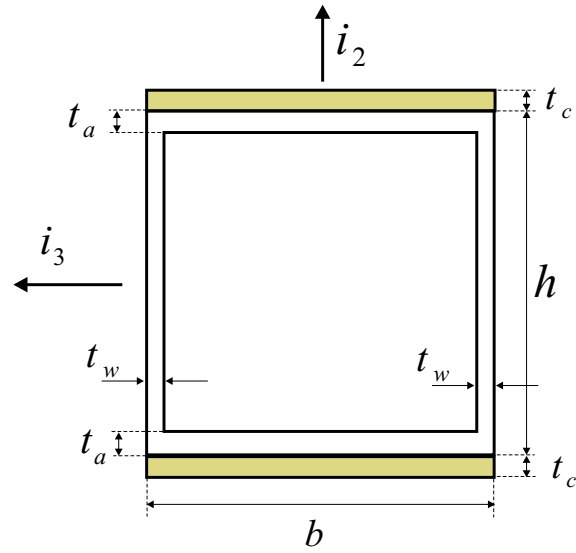


Figure 3.13: Cross-section of a reinforced rectangular box beam.

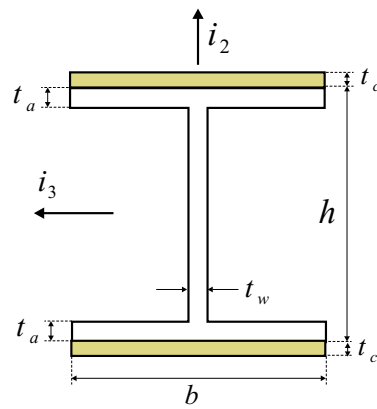


Figure 3.14: Cross-section of a reinforced I beam.

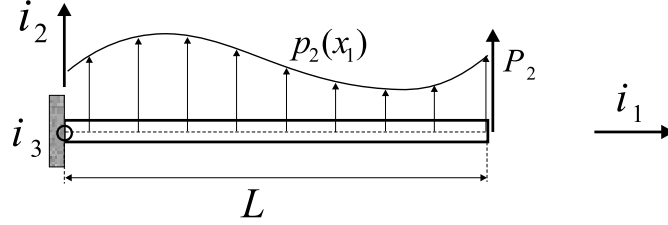


Figure 3.15: Beam subjected to transverse loads.

Problem 3.2

Fig. 3.14 depicts an aluminum I beam of height $h = 0.25 \text{ m}$, width $b = 0.2 \text{ m}$, flange thickness $t_a = 16 \times 10^{-3} \text{ m}$, and web thickness $t_w = 12 \times 10^{-3} \text{ m}$. The beam is reinforced by two layers of unidirectional composite material of thickness $t_c = 5 \times 10^{-3} \text{ m}$. The section is subjected to an axial force $F_1 = 500 \times 10^3 \text{ N}$. The Young's moduli for the aluminum and unidirectional composite are $E_a = 73 \text{ GPa}$ and $E_c = 140 \text{ GPa}$, respectively.

1. Find the distribution of axial stress over the cross-section and sketch it.
2. Find the magnitude and location of the maximum axial stress in the aluminum and composite layer. If the allowable stress for the aluminum and unidirectional composite are $\sigma_a^{\text{allow}} = 400 \text{ MPa}$ and $\sigma_c^{\text{allow}} = 1500 \text{ MPa}$, respectively, find the maximum axial force the section can carry.
3. Sketch the distribution of axial strain over the cross-section.

3.5 Beams subjected to transverse loads

Consider a beam subjected to distributed transverse loads $p_2(x_1)$ as well as a concentrated transverse load P_2 applied at the tip of the beam, for instance, as depicted in fig. 3.15. Under the effect of these applied loads, the beam will bend, creating transverse displacement and curvature fields. Moreover, bending moments, transverse shear forces, and axial and transverse shearing stresses will be generated in the beam.

3.5.1 Kinematic description

To simplify the analysis, we will assume that the plane $(\mathbf{i}_1, \mathbf{i}_2)$ is a plane of symmetry of the structure. Since the loads are applied in this plane of symmetry, the response of the beam will be entirely contained in that plane.

The three Euler-Bernoulli assumptions discussed in the previous sections are made, and furthermore, it seems reasonable to assume that transverse loads only cause transverse displacement and curvature of the section. The displacement field, eq. (3.4) then reduces to

$$u_1(x_1, x_2, x_3) = -(x_2 - x_{2c})U_2'(x_1); \quad (3.38)$$

$$u_2(x_1, x_2, x_3) = U_2(x_1); \quad u_3(x_1, x_2, x_3) = 0. \quad (3.39)$$

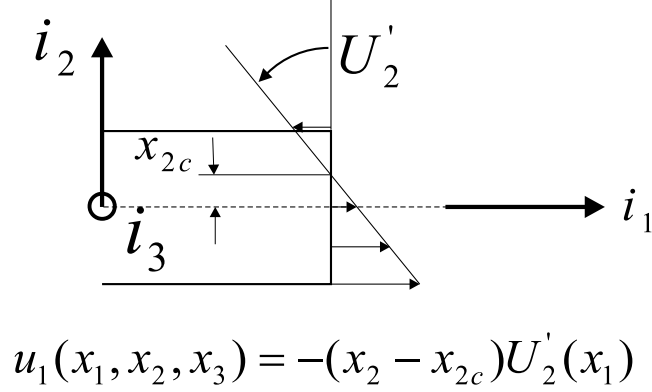


Figure 3.16: Axial displacement distribution.

This displacement field is depicted in fig. 3.16. The only non vanishing strain component is

$$\varepsilon_1(x_1, x_2, x_3) = -(x_2 - x_{2c})\kappa_3(x_1). \quad (3.40)$$

3.5.2 Sectional constitutive law

Beams made of linear elastic materials are considered once more. Since the section does not deform in its own plane, Hooke's law once again reduces to (3.15). The axial stress distribution becomes

$$\sigma_1(x_1, x_2, x_3) = -E(x_2 - x_{2c})\kappa_3(x_1). \quad (3.41)$$

The sectional axial force (3.9) is evaluated as

$$F_1(x_1) = - \left[\int_{\Omega} E (x_2 - x_{2c}) d\Omega \right] \kappa_3(x_1). \quad (3.42)$$

Since the beam is subjected to transverse loads only, this axial force should vanish as can be proved by a simple equilibrium argument. On the other hand, the curvature $\kappa_3(x_1)$ is not zero, and hence, the bracketed term should vanish. Solving for x_{2c} yields

$$x_{2c} = \frac{1}{S} \int_{\Omega} E x_2 d\Omega = \frac{S_2}{S}. \quad (3.43)$$

This defines the coordinate x_{2c} of the centroid of the section.

The centroidal bending moment (3.12) can be evaluated by introducing the axial stress distribution eq. (3.41) to find

$$M_3^c(x_1) = \left[\int_{\Omega} E (x_2 - x_{2c})^2 d\Omega \right] \kappa_3(x_1) = I_{33}^c \kappa_3(x_1), \quad (3.44)$$

where the curvature $\kappa_3(x_1)$ was factored out of the integral over the section. The *centroidal bending stiffness* about the axis $\mathbf{i}_3$ was defined as

$$I_{33}^c = \int_{\Omega} E (x_2 - x_{2c})^2 d\Omega \quad (3.45)$$

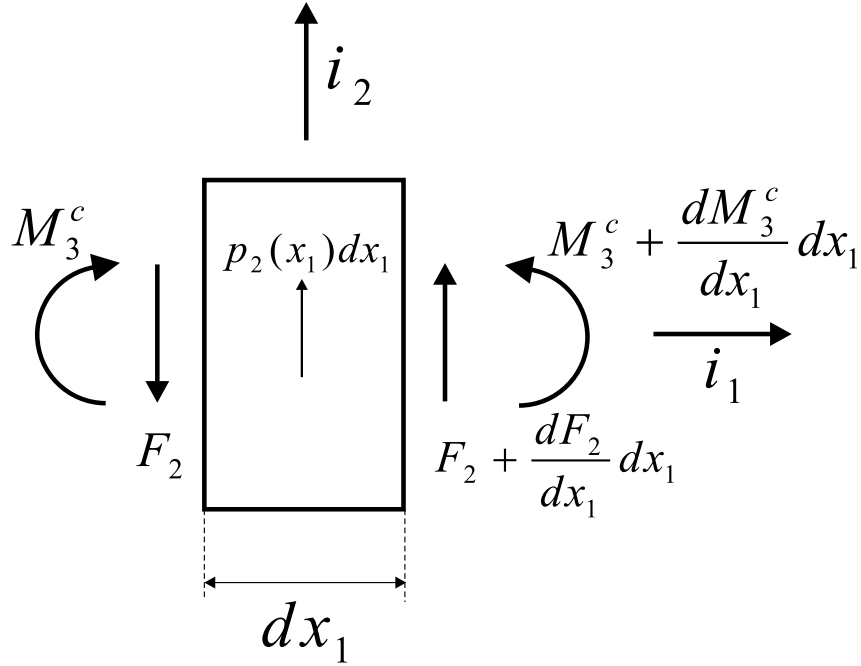


Figure 3.17: Bending moments and shear forces acting on an infinitesimal slice of the beam.

Relationship (3.44) is the constitutive law for the bending behavior of the beam. It expresses the proportionality between the bending moment and the curvature, with a constant of proportionality called the bending stiffness. Both bending moment and bending stiffness are computed about the centroid.

3.5.3 Equilibrium equations

Equilibrium equations are derived to complete the formulation. An infinitesimal slice of the beam of length dx_1 is depicted in fig. 3.17. A free body diagram of this slice yields the following equations of equilibrium

$$\frac{dF_2}{dx_1} = -p_2; \quad \frac{dM_3^c}{dx_1} + F_2 = 0. \quad (3.46)$$

The transverse shearing force F_2 is readily eliminated from these two equilibrium equations to yield the bending moment equilibrium equation

$$\frac{d^2 M_3^c}{dx_1^2} = p_2. \quad (3.47)$$

3.5.4 Governing equations

Finally, the governing equations are found by introducing the bending moment eq. (3.44) into the equation of equilibrium eq. (3.47), and recalling the expression for the curvature

eq. (3.7)

$$\frac{d^2}{dx_1^2} \left[I_{33}^c \frac{d^2 U_2}{dx_1^2} \right] = p_2 \quad (3.48)$$

This fourth order differential equation can be solved for the transverse displacement field $U_2(x_1)$ given the distribution of transverse loading p_2 .

Four boundary conditions are required for the solution of eq. (3.48), two at each end of the beam. Typical boundary conditions are

1. A clamped end restricts both transverse displacement and rotation of the section. Since the rotation of the section and the slope of the beam are equal, eq. (3.3), we find

$$U_2 = 0; \quad \frac{dU_2}{dx_1} = 0. \quad (3.49)$$

2. A simply supported (or pinned) end requires a zero transverse displacement, but the slope of the beam is arbitrary. The pin can not take a bending moment implying a second boundary condition: $M_3^c = 0$. Using eq. (3.44) and the definition of curvature, eq. (3.7) yields

$$U_2 = 0; \quad \frac{d^2 U_2}{dx_1^2} = 0. \quad (3.50)$$

3. At a free (or unloaded) end, both bending moment and shear force must vanish, *i.e.* for eqs. (3.44) and (3.46)

$$\frac{d^2 U_2}{dx_1^2} = 0; \quad \frac{d}{dx_1} \left[I_{33}^c \frac{d^2 U_2}{dx_1^2} \right] = 0. \quad (3.51)$$

4. At an end loaded by a concentrated transverse load P_2 , the shear force must equal the applied load, and the bending moment still vanishes

$$\frac{d}{dx_1} \left[I_{33}^c \frac{d^2 U_2}{dx_1^2} \right] = P_2; \quad \frac{d^2 U_2}{dx_1^2} = 0. \quad (3.52)$$

3.5.5 The sectional bending stiffness

The bending stiffness I_{33}^c of the section characterizes the stiffness of the beam when subjected to bending. If the beam is made of a homogeneous material, Young's modulus is identical at all points on the section and can be factored out of the definition of the centroid, eq. (3.43), to yield

$$x_{2c} = \frac{1}{\Omega} \int_{\Omega} x_2 d\Omega. \quad (3.53)$$

This relationship shows that when the beam is made of a homogeneous material, the centroid is coincident with the center of mass of the cross-section. Similarly, Young's modulus can be factored out of the definition of the bending stiffness (3.45) to yield

$$I_{33}^c = E J_{33}^c, \quad (3.54)$$

where

$$J_{33}^c = \int_{\Omega} (x_2 - x_{2c})^2 d\Omega. \quad (3.55)$$

J_{33}^c is a purely geometric quantity known as the second moment of inertia of the section, computed about the centroid.

On the other hand, if the section is made of several different materials, the bending stiffness must be computed according to (3.45). An important case is that of a rectangular section of width b made of layered materials of different stiffnesses, as depicted in fig. 3.9. Assuming the material to be homogeneous within each layer with a Young modulus $E^{[i]}$ in layer i , the bending stiffness becomes

$$I_{33}^c = \sum_{i=1}^n E^{[i]} \int_{\Omega^{[i]}} (x_2 - x_{2c})^2 d\Omega^{[i]}$$

or

$$I_{33}^c = \sum_{i=1}^n E^{[i]} b \left[\frac{1}{3} [(x_2^{[i+1]})^3 - (x_2^{[i]})^3] - x_{2c} [(x_2^{[i+1]})^2 - (x_2^{[i]})^2] + x_{2c}^2 [x_2^{[i+1]} - x_2^{[i]}] \right].$$

If the centroid is located at the origin of the axes, i.e. if $x_{2c} = 0$, this expression reduces to

$$I_{33}^c = \frac{b}{3} \sum_{i=1}^n E^{[i]} [(x_2^{[i+1]})^3 - (x_2^{[i]})^3]. \quad (3.56)$$

From this expression it is clear that the bending stiffness is a *weighted average* of the Young's moduli of the various layer. The weighting factor $[(x_2^{[i+1]})^3 - (x_2^{[i]})^3]$ strongly biases the average in favor of the outermost layers (for which $x_2^{[i+1]}$ and $x_2^{[i]}$ are large), whereas the layers near the centroid contribute little to the overall bending stiffness ($x_2^{[i+1]}$ and $x_2^{[i]}$ are nearly zero).

3.5.6 The axial stress distribution

The determination the local axial stress σ_1 for a given transverse load p_2 is of primary interest to the designer. This can be readily obtained by eliminating the curvature from eqs. (3.41) and (3.44) to find

$$\sigma_1(x_1, x_2, x_3) = -E (x_2 - x_{2c}) \frac{M_3^c(x_1)}{J_{33}^c}. \quad (3.57)$$

If the beam is made of a homogeneous material, the bending stiffness is given by eqs. (3.54) and (3.57) then reduces to

$$\sigma_1(x_1, x_2, x_3) = -(x_2 - x_{2c}) \frac{M_3^c(x_1)}{J_{33}^c}. \quad (3.58)$$

The axial stress is linearly distributed over the section, and is independent of Young's modulus. For a positive bending moment, the maximum tensile axial stress is found at the point of

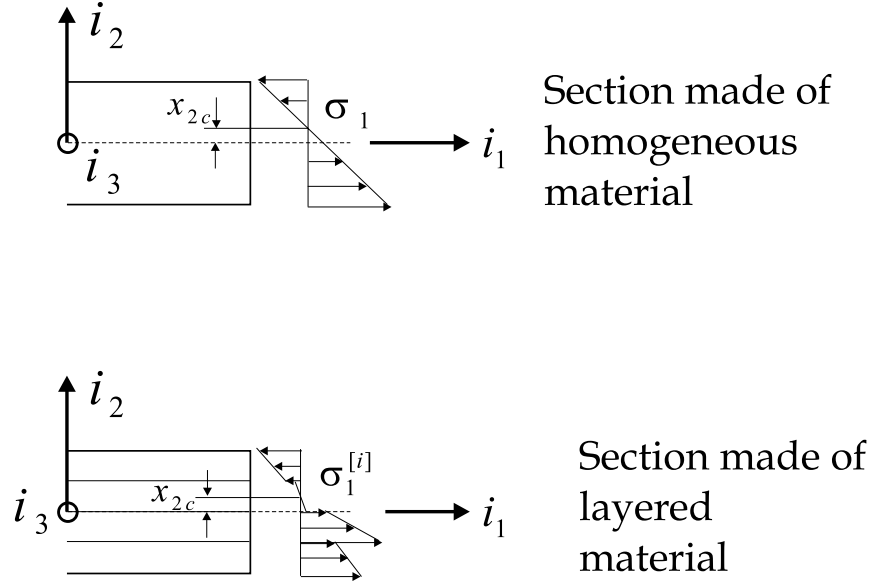


Figure 3.18: Axial stress distributions in homogenous and layered sections.

the section the furthest below from the centroid, *i.e.* at the point for which $v_2^{\min} = x_2 - x_{2c}$ is minimum, whereas the maximum compressive axial stress is found at the point of the section the furthest above from the centroid, *i.e.* at the point for which $v_2^{\max} = x_2 - x_{2c}$ is maximum.

In contrast, the axial stress distribution for sections with various layers will be linear within each layer. Indeed, (3.57) becomes

$$\sigma_1^{[i]}(x_1, x_2, x_3) = -E^{[i]}(x_2 - x_{2c}) \frac{M_3^c(x_1)}{I_{33}^c}. \quad (3.59)$$

Note that according to (3.40) the axial strain distribution is linear over the section, in contrast with the axial stress which is piece-wise linear. The axial stress distribution for homogeneous and layered sections are depicted in fig. 3.18.

Once the local axial stress has been determined, a strength criterion is applied to determine whether the structure can sustain the applied loads. Combining the strength criterion, eq. (??), and eq. (3.57) yields $E|v_2^{\max}| |M_3^c(x_1)| / I_{33}^c \leq \sigma_{\text{allow}}^{\text{comp}}$ and $E|v_2^{\min}| |M_3^c(x_1)| / I_{33}^c \leq \sigma_{\text{allow}}^{\text{tens}}$ for a positive bending moment. The strength criterion becomes

$$\frac{E|v_2^{\max}|}{I_{33}^c} |M_3^{\text{max}}| \leq \sigma_{\text{allow}}^{\text{comp}}, \quad \frac{E|v_2^{\min}|}{I_{33}^c} |M_3^{\text{max}}| \leq \sigma_{\text{allow}}^{\text{tens}}, \quad (3.60)$$

where M_3^{max} is the maximum positive bending moment in the beam and

$$\frac{E|v_2^{\max}|}{I_{33}^c} |M_3^{\text{min}}| \leq \sigma_{\text{allow}}^{\text{tens}}, \quad \frac{E|v_2^{\min}|}{I_{33}^c} |M_3^{\text{min}}| \leq \sigma_{\text{allow}}^{\text{comp}}, \quad (3.61)$$

where M_3^{min} is the minimum negative bending moment in the beam. If the section is such that $|v_2^{\min}| = |v_2^{\max}|$, and/or if the material presents equal tensile and compressive strength, one or more of these four strength criteria might become redundant.

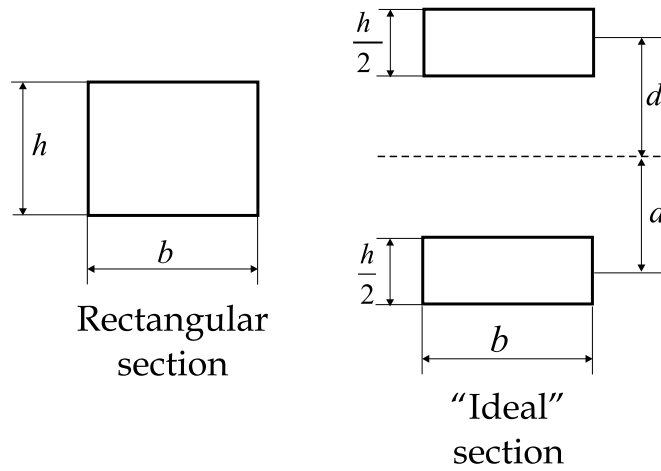


Figure 3.19: A rectangular section, and the ideal section.

Of course, if the section consists of layers made of various materials, the strength of each layer will, in general, be different. Furthermore, the maximum stress does not necessarily occur at the points with the largest distance to the centroid, as illustrated in fig. 3.18. In such case, the axial stress must be computed at the top and bottom locations of each ply, then the strength criterion is applied.

3.5.7 Rational Design of beams under Bending

The axial stress distribution of a beam under bending is given by (3.41). The axial stress clearly vanishes along a straight line passing through the centroid and parallel to the axis $\mathbf{i}_3$. This line is called the neutral axis of the beam. As a result, the material located near the neutral axis carries almost no stresses, and contributes little to the overall load carrying capability of the beam.

A similar conclusion can be drawn from examining the centroidal bending stiffness (3.45): the integrand vanishes along the neutral axis. This means that the material located near the neutral axis contributes little to the bending stiffness of the beam. Clearly, the rational design of a beam under bending calls for the removal of the material located at and near the neutral axis.

Consider first the case of a beam made of homogeneous material. Two different cross-sections are depicted in fig. 3.19: the first section is a rectangle of width b and height h , and the second is composed of two flanges each of width b and height $h/2$ separated by a distance $2d$. Both sections have the same mass per unit span $m = bh\rho$ where ρ is the material density. Clearly, the second section is an idealization since no material connects the two flanges. In practical designs, a thin web would be used to keep the two flanges in their respective positions.

The ratio of the centroidal bending stiffnesses of the two sections denoted I_{ideal} and I_{rect} .

for the ideal and rectangular sections, respectively, is

$$\frac{I_{\text{ideal}}}{I_{\text{rect.}}} = \frac{E \cdot 2 \left[\frac{b(h/2)^3}{12} + \frac{bh}{2} d^2 \right]}{E \left[\frac{bh^3}{12} \right]} = \frac{1}{4} + 12 \left(\frac{d}{h} \right)^2. \quad (3.62)$$

When $d \ll h$ the bending stiffness of the ideal section is much larger than that of the rectangular section. Indeed, for $d/h = 10$, $I_{\text{ideal}}/I_{\text{rect.}} \approx 12(d/h)^2 = 1200$.

The ratio of the maximum axial stress in the two sections denoted $\sigma_{\text{rect.}}^{\text{max}}$ and $\sigma_{\text{ideal}}^{\text{max}}$ for the rectangular and ideal sections, respectively, is found as

$$\frac{\sigma_{\text{rect.}}^{\text{max}}}{\sigma_{\text{ideal}}^{\text{max}}} = \frac{E \frac{h}{2} M_3 \frac{I_{\text{ideal}}}{I_{\text{rect.}} E (d + \frac{h}{4}) M_3}}{\frac{1}{4} + 12 \left(\frac{d}{h} \right)^2} = \frac{\frac{1}{4} + 12 \left(\frac{d}{h} \right)^2}{\frac{1}{2} + 2 \left(\frac{d}{h} \right)}. \quad (3.63)$$

For $d/h = 10$, $\sigma_{\text{rect.}}^{\text{max}}/\sigma_{\text{ideal}}^{\text{max}} \approx 6(d/h) = 60$. In other words, if the same material used for the two sections, the ideal section can carry a 60 times larger bending moment, although the two beams have the same amount of material.

This example shows that the rational design of a beam in bending calls for a section with the largest possible height, and the concentration of all the material as far as possible from the neutral axis. In practical situations the ideal section cannot be used. A web is necessary to connect the two flanges resulting in an I beam design. The maximum height of the section is often limited by other design considerations. Furthermore, as the height of the section increases, it becomes prone to instabilities such as web buckling.

3.5.8 The strain energy

Consider an infinitesimal slice of the beam acted upon by a bending moment M_3^c . Under the effect of this moment, the end cross-sections of the slice rotate of a differential amount $d(dU_2/dx_1)$. If this deformation is proportional to the applied moment, the work dW done by the bending moment as it increases from zero to its present value M_3^c is

$$dW = \frac{1}{2} M_3^c d\left(\frac{dU_2}{dx_1}\right) = \frac{1}{2} M_3^c \frac{d^2 U_2}{dx_1^2} dx_1.$$

This work is given by the area under the bending moment-curvature curve. Introducing the sectional constitutive law, eq. (3.44), and the curvature displacement relationship, eq. (3.7), leads to

$$dW = \frac{1}{2} I_{33}^c \kappa_3^2 dx_1. \quad (3.64)$$

The work done by the bending moment is stored in the slice of the beam in the form of strain energy, and the quantity

$$a(\kappa_3) = \frac{1}{2} I_{33}^c \kappa_3^2, \quad (3.65)$$

is known as the strain energy density function under curvature. Eq. (3.64) expresses the fact that the work done by the bending moment equals the amount of strain energy stored in the

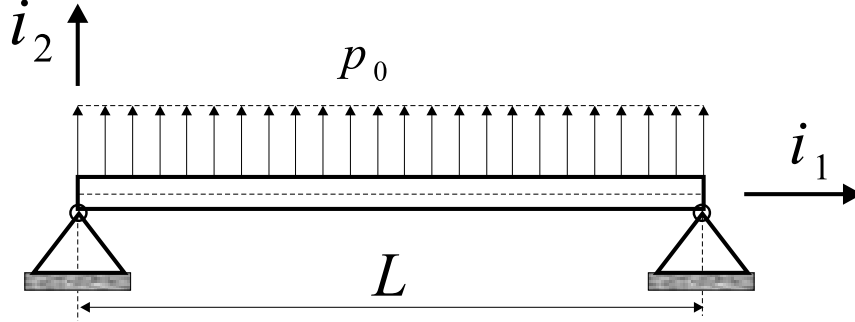


Figure 3.20: Simply supported beam under a uniform transverse load.

slice of the beam. The total work done by the bending moment distribution in the beam is now

$$W = \frac{1}{2} \int_0^L I_{33}^c \kappa_3^2 dx_1 = A(\kappa_3), \quad (3.66)$$

and equal the total amount of strain energy stored in the beam, $A(\kappa_3)$.

Sometimes, it is preferable to express the deformation energy stored in the beam in terms of the bending moment by using eq. (3.44), to find

$$A(\kappa_3) = \int_0^L \frac{M_3^c{}^2}{2I_{33}^c} dx_1 = B(M_3^c). \quad (3.67)$$

$b(M_3^c) = M_3^c{}^2/2I_{33}^c$ is known as the *stress energy density function*. $B(M_3^c)$ is the total deformation energy stored in the beam expressed in terms of the bending moment, also called *complementary energy*.

3.5.9 Examples

Example 1: Simply supported beam under a uniform load

Consider a simply supported, uniform beam of length L subjected to a uniform transverse loading $p_2(x_1) = p_0$, as depicted in fig. 3.20. For this problem, the governing equation, eq. (3.48) becomes

$$I_{33}^c \frac{d^4 U_2}{dx_1^4} = p_0;$$

$$@x_1 = 0, \quad U_2 = I_{33}^c \frac{d^2 U_2}{dx_1^2} = 0; \quad @x_1 = L, \quad U_2 = I_{33}^c \frac{d^2 U_2}{dx_1^2} = 0,$$

where the boundary conditions expresses the vanishing of the transverse displacement and bending moment at the two end supports. The solution of this differential equation is

$$U_2 = \frac{p_0 L^4}{24 I_{33}^c} \left[\left(\frac{x_1}{L} \right) - 2 \left(\frac{x_1}{L} \right)^3 + \left(\frac{x_1}{L} \right)^4 \right] \quad (3.68)$$

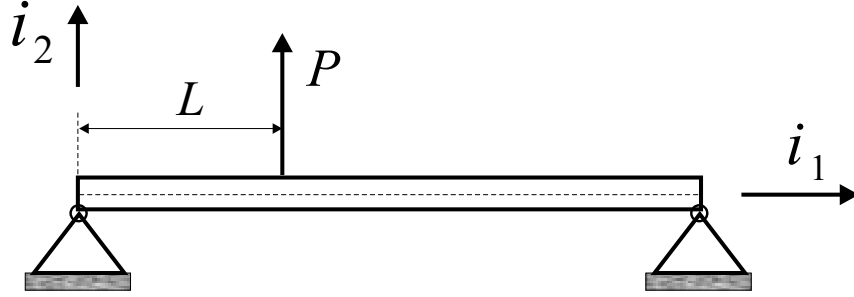


Figure 3.21: Simply supported beam with one concentrated load.

The bending moment distribution then follows from eq. (3.44)

$$M_3^c = \frac{p_0 L^2}{2} \left[\left(\frac{x_1}{L} \right) - 2 \left(\frac{x_1}{L} \right)^2 \right]. \quad (3.69)$$

As expected, the bending moment is maximum at mid-span, $M_{3\max}^c = p_0 L^2 / 8$. The axial stress at any point in the beam can then be obtained from the formulas developed in section 3.5.6.

Example 2: Simply supported beam with concentrated load

Consider now a simply supported, uniform beam of length L subjected to a concentrated load P acting at a distance αL from the end supports, as depicted in fig. 3.21. The bending moment distribution can be readily obtained from simple equilibrium considerations. A free body diagram of the entire beam reveals that the reaction forces are $(1 - \alpha)P$ and αP , at the left and right end supports, respectively. Next, the equilibrium condition for a free body diagram of a piece of the beam extending from the left support to a location $0 < x_1 < \alpha L$ yields

$$M_3^c = -(1 - \alpha)P x_1 = -(1 - \alpha)PL\eta,$$

where $\eta = x_1 / L$ is the nondimensional variable along the span of the beam. Similarly, a free body diagram of a piece of the beam extending from the left support to a location $x_1 > \alpha L$ leads to

$$M_3^c = -\alpha P(L - x_1) = -\alpha PL(1 - \eta).$$

The bending moment curvature relationship, eq. (3.44), is now integrated twice to yield

$$U_2(\eta) = \frac{PL^3}{I_{33}^c} \begin{cases} -(1 - \alpha)\frac{\eta^3}{6} + C_1\eta + C_2; & 0 \leq \eta \leq \alpha \\ -\alpha\left(\frac{\eta^2}{2} - \frac{\eta^3}{6}\right) + C_3\eta + C_4; & \alpha < \eta \leq 1 \end{cases},$$

where C_1 , C_2 , C_3 and C_4 are four integration constants to be evaluated from the boundary conditions. $U_2(0) = U_2(1) = 0$ because the beam is simply supported at the two ends, and

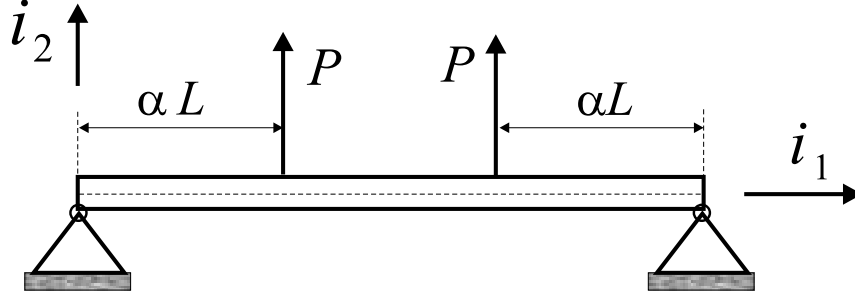


Figure 3.22: Simply supported beam with two concentrated loads.

furthermore, at $\eta = \alpha$, both displacement and slope of the beam must be continuous. These four conditions are sufficient to determine the four integration constants to yield

$$U_2(\eta) = \frac{PL^3}{6I_{33}^c} \begin{cases} -(1-\alpha)\eta^3 + \alpha(2-\alpha)(1-\alpha)\eta; & 0 \leq \eta \leq \alpha \\ \alpha(\eta^3 - 3\eta^2) + \alpha(2+\alpha^2)\eta - \alpha^3; & \alpha < \eta \leq 1 \end{cases}, \quad (3.70)$$

The bending moment distribution then follows

$$M_3^c(x_1) = PL \begin{cases} -(1-\alpha)\eta; & 0 \leq \eta \leq \alpha \\ -\alpha(1-\eta); & \alpha < \eta \leq 1 \end{cases}. \quad (3.71)$$

Finally, the shear force distribution reflects the values of the reactions forces at the end supports

$$F_2^c(x_1) = P \begin{cases} (1-\alpha); & 0 \leq \eta \leq \alpha \\ -\alpha; & \alpha < \eta \leq 1 \end{cases}. \quad (3.72)$$

At $\eta = \alpha$, the shear force presents a discontinuity that is equal to the applied concentrated load at that point.

Example 3: Simply supported beam with concentrated loads

Consider now a simply supported, uniform beam of length L subjected to two concentrated loads acting at a distance αL from the end supports, as depicted in fig. 3.22. It is clear that this problem presents discontinuities in the transverse shear force for distribution due the presence of the concentrated loads. The bending moment distribution can be readily obtained from simple equilibrium considerations. A free body diagram of the entire beam reveals that the reaction forces at the end supports must each equal P . Next, the equilibrium condition for a free body diagram of a piece of the beam extending from the left support to a location $0 < x_1 < \alpha L$ yields

$$M_3^c = -Px_1.$$

Similarly, a free body diagram of a piece of the beam extending from the left support to a location $\alpha L < x_1 < L/2$ leads to

$$M_3^c = P(x_1 - \alpha L) - Px_1 = -P\alpha L.$$

The bending moment curvature relationship, eq. (3.44), is now integrated twice to yield

$$I_{33}^c U_2(x_1) = \begin{cases} -\frac{Px_1^3}{6} + C_1x_1 + C_2; & 0 \leq x_1 \leq \alpha L \\ -\alpha L \frac{Px_1^2}{2} + C_3x_1 + C_4; & \alpha L < x_1 \leq L/2 \end{cases},$$

where C_1 , C_2 , C_3 and C_4 are four integration constants. Two of these constants can be computed from the boundary conditions: the simple support at the root, $U_2(0) = 0$, and the symmetry condition at mid-span, $dU_2/dx_1(L/2) = 0$. The other two integration constants are found by imposing the continuity of the displacement and of its derivative at $x_1 = \alpha L$. The transverse displacement distribution becomes

$$U_2(x_1) = \frac{PL^3}{6I_{33}^c} \begin{cases} \left[3(\alpha - \alpha^2)\left(\frac{x_1}{L}\right) - \left(\frac{x_1}{L}\right)^3 \right]; & 0 \leq x_1 \leq \alpha L \\ \left[-\alpha^3 + 3\alpha\left(\frac{x_1}{L}\right) - 3\alpha\left(\frac{x_1}{L}\right)^2 \right]; & \alpha L < x_1 \leq L/2 \end{cases}.$$

The bending moment distribution then follows

$$M_3^c(x_1) = PL \begin{cases} -\left(\frac{x_1}{L}\right); & 0 \leq x_1 \leq \alpha L \\ -\alpha; & \alpha L < x_1 \leq L/2 \end{cases}.$$

This result can be easily verified: the portion of the beam between the two concentrated loads is subjected to a constant bending moment αLP , and hence, the shear force vanishes. At $x_1 = \alpha L$ the shear force jumps from $F_2 = P$, immediately to the left, to $F_2 = 0$, immediately to the right. This jump is of course equal to the force applied at that point.

The problem could also be solved using the governing differential equation, eq. (3.48). However, two separate problems should be solved, one for $0 \leq x_1 \leq \alpha L$, and another for $\alpha L < x_1 \leq L/2$. The integration of these two, fourth order equations will generate a total of eight integration constants to be computed from four boundary conditions, and four continuity conditions at $x_1 = \alpha L$. The boundary conditions at the root are $U_2(0) = 0$ and $d^2U_2/dx_1^2(0) = 0$, corresponding to the vanishing of the displacement and bending moment; at mid-span $dU_2/dx_1(L/2) = 0$ and $d^3U_2/dx_1^3(L/2) = 0$, corresponding to the symmetry conditions of vanishing slope and shear force. At $x_1 = \alpha L$ continuity of displacement, slope and bending moment is required together with the enforcement of a jump in shear force. The approach used above is clearly less laborious.

Example 3: Simply supported beam with two elastic spring supports

A simply supported beam of span L is supported by two spring of stiffness k located at stations $x_1 = \alpha L$ and $(1 - \alpha)L$, and is subjected to a uniform transverse loading p_0 , as depicted in fig. 3.23. At first, the springs are replaced by two unknown forces F acting downward at $x_1 = L/3$ and $2L/3$. The transverse displacement distribution is then readily obtained by superposing the displacement field for a beam under uniform loading (example 1) and that of a beam under two concentrated forces (example 3)

$$U_2(\eta) = \begin{cases} \frac{p_0L^4}{24I_{33}^c} [\eta - 2\eta^3 + \eta^4] - \frac{FL^3}{6I_{33}^c} [3(\alpha - \alpha^2)\eta - \eta^3]; & 0 \leq \eta \leq \alpha \\ \frac{p_0L^4}{24I_{33}^c} [\eta - 2\eta^3 + \eta^4] - \frac{FL^3}{6I_{33}^c} [-\alpha^3 + 3\alpha\eta - 3\alpha\eta^2]; & \alpha < \eta \leq 1/2 \end{cases},$$

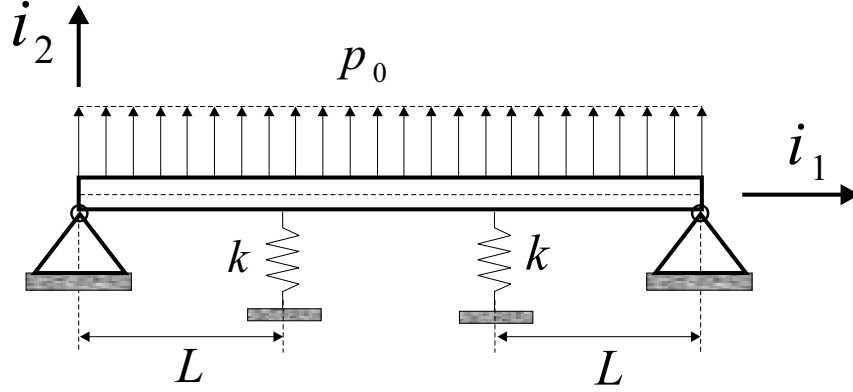


Figure 3.23: Simply supported beam with two elastic spring supports.

where $\eta = x_1/L$ if the nondimensional coordinate along the beam axis. The unknown force F acts on the spring, and hence $F = kU_2(\alpha)$. Introducing the above expression for $U_2(\alpha L)$ provides an additional equation to evaluate the unknown for F

$$\frac{F}{p_0 L} = \frac{1}{4} \frac{k^*(\alpha^4 - 2\alpha^3 + \alpha)}{6 + k^*(3\alpha^2 - 4\alpha^3)} = \frac{1}{4} \mu, \quad (3.73)$$

where $k^* = kL^3/I_{33}^c$ is the nondimensional spring stiffness. The transverse displacement distribution now becomes

$$U_2(\eta) = \frac{p_0 L^4}{24 I_{33}^c} \begin{cases} \eta^4 - (2 - \mu) \eta^3 + [1 - 3\mu(\alpha - \alpha^2)] \eta; & 0 \leq \eta \leq \alpha \\ \eta^4 - 2\eta^3 + 3\alpha\mu \eta^2 + (1 - 3\alpha\mu) \eta + \alpha^3\mu; & \alpha < \eta \leq 1/2 \end{cases} \quad (3.74)$$

The bending moment distribution then follows from eq. (3.44)

$$M_3^c(\eta) = \frac{p_0 L^2}{2} \begin{cases} \eta^2 - (1 - \mu/2) \eta; & 0 \leq \eta \leq \alpha \\ \eta^2 - \eta + \alpha\mu/2; & \alpha < \eta \leq 1/2 \end{cases} \quad (3.75)$$

Finally, the shear force distribution is obtained from eq. (3.46)

$$F_2(\eta) = \frac{p_0 L}{2} \begin{cases} 2\eta - (1 - \mu/2); & 0 \leq \eta \leq \alpha \\ 2\eta - 1; & \alpha < \eta \leq 1/2 \end{cases} \quad (3.76)$$

As anticipated, this distribution presents a discontinuity due to the concentrated forces the springs apply to the beam. Indeed, the shear forces immediately to the left and right of the spring located at $\eta = \alpha$, denoted F_{2l} and F_{2r} , respectively are such that $F_{2r} - F_{2l} = \mu p_0 L/4 = F$.

Example 4: Simply supported beam on an elastic foundation

The last example deals with a simply supported beam of length L subjected to a uniform transverse load p_0 , as depicted in fig. 3.24. The beam is supported by an elastic foundation

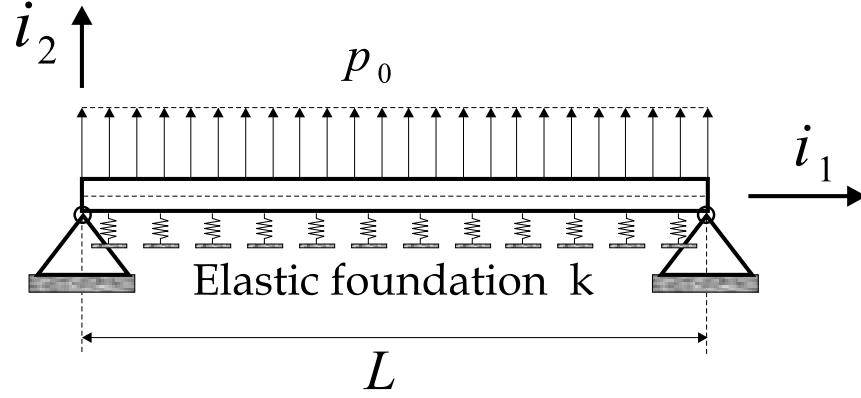


Figure 3.24: Simply supported beam on an elastic foundation.

of distributed stiffness k . For this problem, the governing equation is

$$I_{33}^c \frac{d^4 U_2}{dx_1^4} = p_1(x_1);$$

$$@x_1 = 0, \quad U_2 = I_{33}^c \frac{d^2 U_2}{dx_1^2} = 0; \quad @x_1 = L, \quad U_2 = I_{33}^c \frac{d^2 U_2}{dx_1^2} = 0.$$

The total applied load $p_1(x_1) = p_0 - k U_2(x_1)$, where the first term accounts for the applied load, and the second corresponds to the distributed restoring force of the elastic foundation. The governing equation can be recast as

$$U_2'''' + \frac{kL^4}{I_{33}^c} U_2 = \frac{p_0 L^4}{I_{33}^c};$$

$$@\eta = 0, \quad U_2 = U_2'' = 0; \quad @\eta = 1, \quad U_2 = U_2'' = 0,$$

where $\eta = x_1/L$ is the nondimensional coordinate along the beam, and $()'$ denotes a derivative with respect to η . The characteristic equation corresponding to the governing homogenous differential equations is $z^4 + kL^4/I_{33}^c = 0$ with characteristic roots

$$z = \pm \sqrt[4]{\frac{kL^4}{4I_{33}^c}} (1 \pm i) = \pm \alpha (1 \pm i),$$

where $i = \sqrt{-1}$. The general solution of the differential equation now writes

$$U_2(\eta) = A e^{\alpha(1+i)\eta} + B e^{\alpha(1-i)\eta} + C e^{-\alpha(1+i)\eta} + D e^{-\alpha(1-i)\eta} + \frac{p_0 L^4}{\alpha^4 I_{33}^c},$$

where A , B , C , and D are four integration constants. Using the relationships between the exponential function with imaginary and real exponents, and the trigonometric and hyperbolic functions, respectively, this expression can be written as

$$U_2 = \cosh \alpha \eta (C_1 \cos \alpha \eta + C_2 \sin \alpha \eta) + \sinh \alpha \eta (C_3 \cos \alpha \eta + C_4 \sin \alpha \eta) + \frac{p_0}{k},$$

where C_1 , C_2 , C_3 , and C_4 form a different set of integration constants that can be evaluated from the four boundary conditions to find

$$\frac{I_{33}^c U_2}{p_0 L^4} = \frac{1}{k^*} [1 - \cosh \alpha \cos \alpha \eta + \mu (\sinh \alpha \sinh \alpha \eta \cos \alpha \eta - \sin \alpha \cosh \alpha \eta \sin \alpha \eta)], \quad (3.77)$$

where $\mu = (\cosh \alpha - \cos \alpha) / (\sin^2 \alpha + \sinh^2 \alpha)$, and $k^* = kL^4 / I_{33}^c$ is the nondimensional elastic foundation stiffness. The bending moment distribution now follows from eq. (3.44)

$$\frac{M_3^c}{p_0 L^2} = \frac{2\alpha^2}{k^*} [1 - \cosh \alpha \cos \alpha \eta + \mu (\sinh \alpha \sinh \alpha \eta \cos \alpha \eta - \sin \alpha \cosh \alpha \eta \sin \alpha \eta)]. \quad (3.78)$$

This example clearly demonstrate that the solution of beam problems can rapidly become quite difficult. The simple addition of an elastic foundation to the beam significantly complicates the solution process that often becomes unmanageable when realistic structural problems are considered.

3.5.10 Problems

Problem 3.3

Fig. 3.14 depicts an aluminum I beam of height $h = 0.25 \text{ m}$, width $b = 0.2 \text{ m}$, flange thickness $t_a = 16 \times 10^{-3} \text{ m}$, and web thickness $t_w = 12 \times 10^{-3} \text{ m}$. The beam is reinforced by two layers of unidirectional composite material of thickness $t_c = 5 \times 10^{-3} \text{ m}$. The section is subjected to a bending moment $M_3^c = 200 \times 10^3 \text{ Nm}$. The Young's moduli for the aluminum and unidirectional composite are $E_a = 73 \text{ GPa}$ and $E_c = 140 \text{ GPa}$, respectively.

1. Find the distribution of axial stress over the cross-section and sketch it.
2. Find the magnitude and location of the maximum axial stress in the aluminum and composite layer. If the allowable stress for the aluminum and unidirectional composite are $\sigma_a^{\text{allow}} = 400 \text{ MPa}$ and $\sigma_c^{\text{allow}} = 1500 \text{ MPa}$, respectively, find the maximum bending moment the section can carry.
3. Sketch the distribution of axial strain over the cross-section.

Problem 3.4

Fig. 3.13 depicts an aluminum rectangular box beam of height $h = 0.30 \text{ m}$, width $b = 0.15 \text{ m}$, flange thickness $t_a = 12 \times 10^{-3} \text{ m}$, and web thickness $t_w = 5 \times 10^{-3} \text{ m}$. The beam is reinforced by two layers of unidirectional composite material of thickness $t_c = 4 \times 10^{-3} \text{ m}$. The section is subjected to an axial load $F_1 = 600 \times 10^3 \text{ N}$, and bending moments $M_2^c = 50 \times 10^3 \text{ Nm}$, $M_3^c = 120 \times 10^3 \text{ Nm}$. The Young's moduli for the aluminum and unidirectional composite are $E_a = 73 \text{ GPa}$ and $E_c = 140 \text{ GPa}$, respectively.

1. Find the distribution of axial stress over the cross-section and sketch it.
2. Find the magnitude and location of the maximum axial stress in the aluminum and composite layer. If the allowable stress for the aluminum and unidirectional composite are $\sigma_a^{\text{allow}} = 400 \text{ MPa}$ and $\sigma_c^{\text{allow}} = 1500 \text{ MPa}$, respectively, find the maximum loading the section can carry.
3. Sketch the distribution of axial strain over the cross-section.

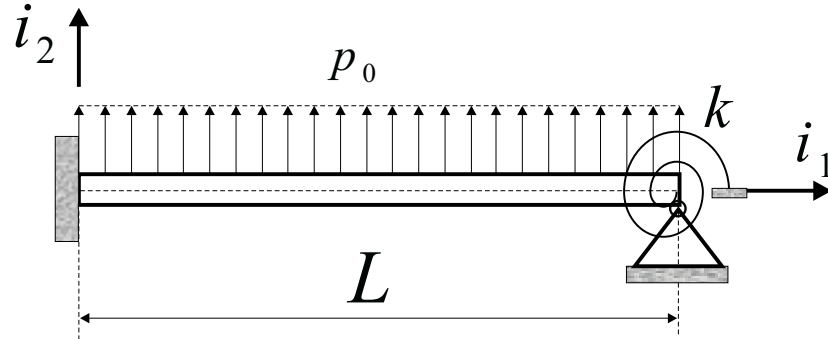
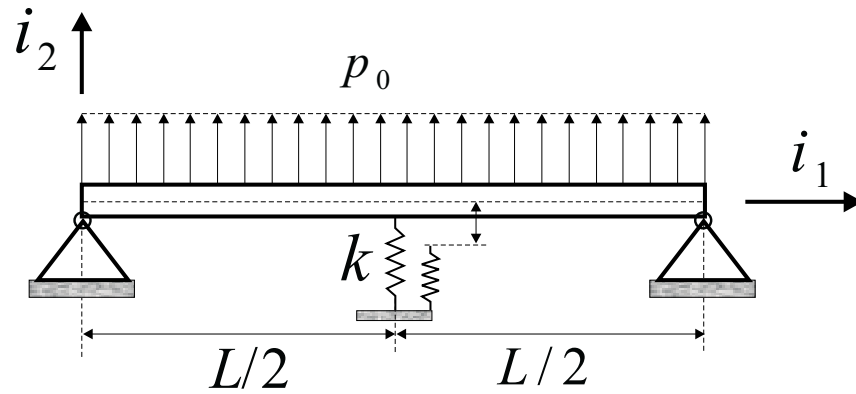


Figure 3.25: Cantilevered beam with tip support and torsional spring.

Figure 3.26: Simply supported beam with mid-span spring featuring a clearance Δ .**Problem 3.5**

Consider a cantilevered beam of span L with a tip support and a torsional spring of stiffness constant k , as depicted in fig. 3.25.

1. Find and plot the transverse displacement distribution of this beam under a uniform transverse load p_0 .
2. Find and plot the distribution of bending moment in the beam.
3. Find the location and magnitude of the maximum bending moment in the beam as a function of $k^* = kL/I_{33}^c$.
4. Discuss your results when $k^* \rightarrow 0$ and $k^* \rightarrow \infty$.

Problem 3.6

Consider the uniform beam with simply supported ends as depicted in fig. 3.26. A spring of stiffness constant k is acting at mid-span ($k^* = kL^3/I_{33}^c$). A uniform load p_0 is acting over the beam span

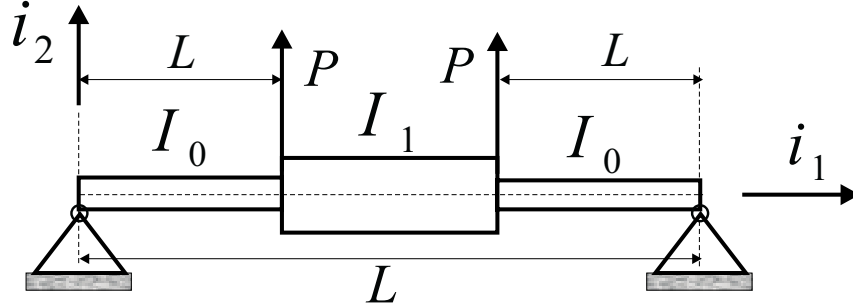


Figure 3.27: Simply supported beam with varying bending stiffness.

of length L . The unstressed length of the spring is such that it fall a distance Δ short of reaching the beam ($\Delta^* = \Delta I_{33}^c / (p_0 L^4)$).

1. Find and plot the transverse displacement distribution for this beam.
2. Find and plot the corresponding distribution of bending moment.
3. What value of Δ^* that will minimize the maximum bending moment in the beam.

Hint: replace the spring by an unknown force F acting at mid-span. This force can then be evaluated by equating the beam mid-span displacement with that of the spring.

Problem 3.7

A simply supported beam of span L is subjected to forces of magnitude P located at stations $x_1 = \alpha L$ and $(1 - \alpha)L$, as depicted in fig. 3.27. The beam has a bending stiffness I_0 and is reinforced in its central portion where its bending stiffness is I_1 .

1. Find the exact solution of the problem from the solution of the governing differential equation and associated boundary conditions.
2. Plot the distribution of transverse displacement for the beam. Use $I_1/I_0 = 2$ and $\alpha = 0.3$.
3. Plot the distribution of bending moment for the beam.
4. Plot the distribution of shear force for the beam.

Problem 3.8

The lower beam depicted in fig. 3.28 is of length $2L$ and is simply supported at both ends. The upper beam of length $L + a$ is cantilevered at the root, supported by the lower beam at point A, and subjected to a uniform transverse loading p_0 . Both upper and lower beams have a uniform bending stiffness I_0 .

1. Find the exact solution for the transverse deflection of the lower beam under a mid-span concentrated load. Show that the lower beam can be replaced by a concentrated spring of stiffness constant

$$k_{\text{eq}} = \frac{6I_0}{L^3}.$$

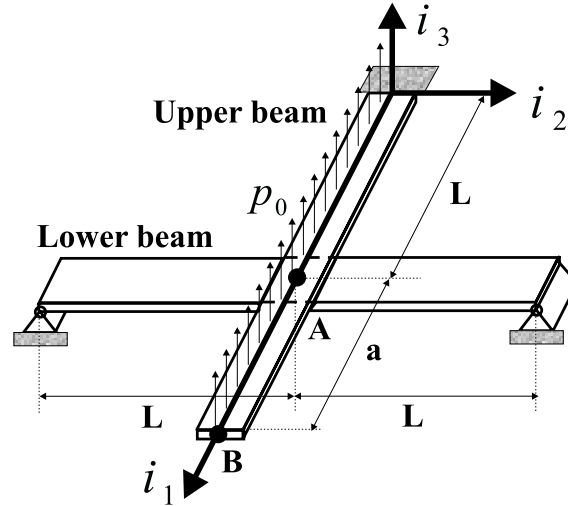


Figure 3.28: Two beam assembly under transverse load.

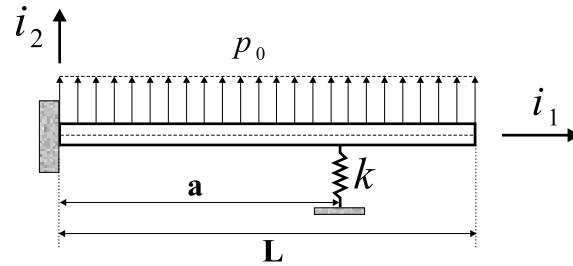


Figure 3.29: Cantilevered beam with concentrated spring.

2. Find the exact solution of the problem from the solutions of the governing differential equations and associated boundary conditions for both upper and lower beams. Replace the interaction between the beams by a force X , yet unknown. The magnitude of this force is found by equating the displacements of the upper and lower beams at point A.
3. Plot the distribution of transverse displacement for both beams. Use $L/a = 2$.
4. Plot the distribution of bending moment for both beams.
5. Plot the distribution of shear force for both beams.

Problem 3.9

The cantilevered beam depicted in fig. 3.29 is of length L , uniform bending stiffness I_{33} , and is subjected to a uniform distributed load p_0 . A concentrated spring of stiffness constant k is connected to the beam at a distance a from its root.

1. Find the exact solution of the problem from the solution of the governing differential equations and associated boundary conditions for the cantilevered beam. It will be necessary to solve

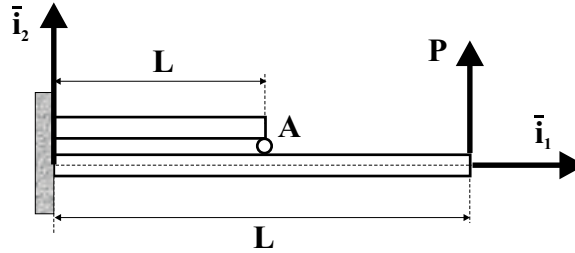


Figure 3.30: Cantilevered beam with intermediate elastic support.

the problem in two parts, for the segments to the right and left of the spring, then applying continuity conditions at this point. It will be convenient to define the non-dimensional spring constant

$$k^* = \frac{ka^3}{I_{33}}$$

2. Plot the distribution of transverse displacement for both beams. Use $L/a = 3$ and $k^* = 100$.
3. Plot the distribution of bending moment for both beams.
4. Plot the distribution of shear force for both beams.
5. Find the value of k^* that will minimize the maximum bending moment in the beam.

Problem 3.10

The cantilevered beam depicted in fig. 3.30 is subjected to a tip load P . The tip of a second cantilevered beam contacts the first at point **A**. The lower and upper beams have a uniform bending stiffness I_{33} and are of length L and αL , respectively.

1. Find the displacement fields for the two beams by solving of the governing differential equations and associated boundary conditions.
2. Plot the distribution of transverse displacement, bending moment, and shear force for both beams. Use $\alpha = 1/2$.
3. Find the magnitude and location of the maximum bending moment in the beams. Plot these quantities as a function of α .

Problem 3.11

The two simply supported beam shown in fig. 3.31 are connected by an intermediate roller located a distance αL from the left support. The upper beam is subjected to a uniform loading p_0 . Both beams have a uniform bending stiffness I_{33} .

1. Find the displacement fields for the two beams by solving of the governing differential equations and associated boundary conditions.
2. Plot the distribution of transverse displacement, bending moment, and shear force for both beams. Use $\alpha = 1/3$.

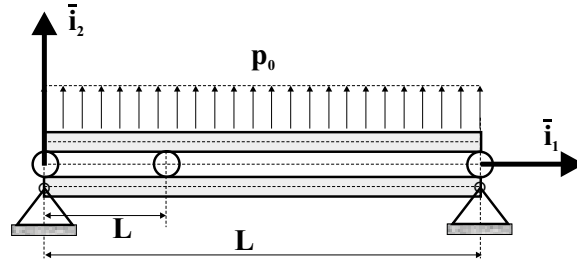


Figure 3.31: Superposed simply supported beams.

3. Find the magnitude and location of the maximum bending moment in the beams. Plot these quantities as a function of α .

3.6 Variational formulation of beam problems

The above sections have described the classical approach to beam theory which involves three major ingredients: a differential equation of equilibrium, a sectional constitutive law, and a strain displacement relationship. The differential equations of equilibrium were derived from Newton's law stating that the sum of the forces acting on a differential element of the system should vanish. The sectional constitutive laws were obtained by integrating the material constitutive laws expressed on a differential element of the material over the beam cross-section, resulting in relationships between stress resultants and sectional strains. Finally the strain displacement relationship stemmed from the kinematic assumptions underlying Euler-Bernoulli beam theory.

Solutions of these governing differential equations provide an exact solution of Euler-Bernoulli beam problem. In practical situations, structures are subjected to complex loading conditions, and the sectional stiffnesses, such as axial or bending stiffnesses, vary drastically over the beam span. Consequently, the governing differential equations become increasingly difficult to solve, and in most practical situations, it is nearly impossible to find closed form solutions.

In this section, variational formulations are introduced for beam problems. In these formulations, the differential equations of equilibrium are replaced by an integral, or *weak statement*, and the one to one equivalence of the two approaches is established. In view of this equivalence, it is clear that exact solutions of a problem are equally difficult to obtain using either approaches. However, variational formulations lead to a systematic procedure for obtaining *approximate solutions* to complex problems. In fact, variational formulations are the basis for the *finite element method* (FEM) which has become the tool of choice for the analysis of complex aerospace structures. Although approximate, FEM solutions often are very accurate and reliable.

3.6.1 The weak form for beams under axial loads

In the previous sections, the differential equation of equilibrium of a beam under axial loads, eq. (3.19), was derived. This equation holds at all points over the span of the beam, *i.e.* for $x_1 \in [0, L]$. At the loaded end of the beam, equilibrium requires the internal force to equal the externally applied load, $F_1(L) = P_1$. These two equilibrium requirements are repeated here

$$\begin{aligned} \frac{dF_1}{dx_1} &= -p_1, & x_1 &\in [0, L]; \\ F_1 &= P_1, & x_1 &= L. \end{aligned} \quad (3.79)$$

These equations are known as the *strong statement of equilibrium*. From this statement, the following integral can be constructed

$$\int_0^L -w(x_1) \left[\frac{dF_1}{dx_1} + p_1 \right] dx_1 + [w(L) (F_1 - P_1)]_L = 0. \quad (3.80)$$

Note that if the beam is in equilibrium eqs. (3.79) hold, and eq. (3.80) is satisfied *for all arbitrary functions* $w(x_1)$. Next, an integration by parts is performed on the first term appearing in the integral

$$\int_0^L -w(x_1) \frac{dF_1}{dx_1} dx_1 = \int_0^L \frac{dw}{dx_1} F_1 dx_1 - [wF_1]_0^L.$$

Introducing this result into eq. (3.80) leads to

$$\int_0^L \frac{dw}{dx_1} F_1 dx_1 - \int_0^L w p_1 dx_1 - w(L)P_1 + w(0)F_1(0) = 0. \quad (3.81)$$

Since $w(x_1)$ is an entirely arbitrary function, $w(0) = 0$ can be selected and this choice yields

$$\begin{aligned} \int_0^L \frac{dw}{dx_1} F_1 dx_1 - \int_0^L w p_1 dx_1 - w(L)P_1 &= 0, \\ \text{for all arbitrary } w(x_1) \text{ such that } w(0) &= 0. \end{aligned} \quad (3.82)$$

This integral is known as the *weak statement of equilibrium*. It is now clear that the strong statement, eq. (3.79), implies the weak statement, eq. (3.82). On the other hand, it is easily shown that the weak statement implies the strong statement. Indeed, the weak statement implies eq. (3.81), which in turns implies eq. (3.80) by reversing the integration by parts process. Finally, the strong statement of equilibrium is implied by eq. (3.80) because this equation must hold for all arbitrary choices of $w(x_1)$.

In summary, the strong statement, eq. (3.79), and the weak statement, eq. (3.82), are two entirely equivalent statements that both express the equilibrium conditions for the beam under axial loads. The arbitrary functions $w(x_1)$ are often called *weighting functions* or *test functions*.

If the beam were to be clamped at the right end and subjected to an axial load at the left end, the above reasoning would still hold. However, the last term of the weak statement would read $w(0)P_1$, and the integral would vanish for all $w(x_1)$ such that $w(L) = 0$. More generally, *geometric boundary conditions* are defined as those boundary conditions that restrict allowable displacements, such as the clamping of the beam at a point. The integral appearing in the weak statement should vanish for all $w(x_1)$ that satisfy the *geometric boundary conditions*.

3.6.2 Examples

Example 1: Beam under a uniform axial load

Consider the uniform, clamped beam of length L depicted in fig. 3.6 subjected to a uniform axial loading $p_1(x_1) = p_0$. The solution of this problem presented in section 3.4.8 was based on the strong statement of equilibrium. Note that this equation of equilibrium is not sufficient to lead to the solution of the problem; it must be complemented by a constitutive law and a strain displacement relationship. The solution of the same problem will now be derived with the help of the weak statement of equilibrium, eq. (3.82). Here again, this statement must be complemented by a constitutive law, eq. (3.17), and a strain displacement relationship, eq. (3.7) to yield

$$\int_0^L \frac{dw}{dx_1} S \frac{dU_1}{dx_1} dx_1 - \int_0^L w p_0 dx_1 = 0.$$

This integral must vanish for all arbitrary functions $w(x_1)$ such that $w(0) = 0$. At this point, the following forms will be assumed for the solution $U_1(x_1)$ and test functions $w(x_1)$

$$U_1(x_1) = ax_1 + bx_1^2; \quad w(x_1) = \alpha x_1 + \beta x_1^2, \quad (3.83)$$

where a and b are two unknown coefficients, and the coefficients α and β are arbitrary since $w(x_1)$ is itself an arbitrary function. Note that for this choice, the solution satisfies the geometric boundary condition $U_1(0) = 0$, and the test functions satisfy the conditions $w(0) = 0$. With these approximations, the weak statement becomes

$$\int_0^L S(\alpha + 2\beta x_1)(a + 2bx_1) dx_1 - \int_0^L p_0(\alpha x_1 + \beta x_1^2) dx_1 = 0.$$

After expansion and integration we find

$$S \left[\alpha a L + 2\alpha b \frac{L^2}{2} + 2\beta a \frac{L^2}{2} + 4\beta b \frac{L^3}{3} \right] - p_0 \left(\alpha \frac{L^2}{2} + \beta \frac{L^3}{3} \right) = 0. \quad (3.84)$$

It is customary to write this equation in a matrix form

$$\begin{bmatrix} \alpha & \beta \end{bmatrix} SL \begin{bmatrix} 1 & L \\ L & 4L^2/3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} \alpha & \beta \end{bmatrix} p_0 L^2 \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} = 0. \quad (3.85)$$

The two by two matrix of stiffness coefficients is called the *stiffness matrix* K , and the array of loading coefficients the *load vector* $\mathbf{Q}$

$$K = SL \begin{bmatrix} 1 & L \\ L & 4L^2/3 \end{bmatrix}, \quad \mathbf{Q} = p_0 L^2 \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}. \quad (3.86)$$

The array of displacement coefficients is called the *solution vector* $\mathbf{q}$ and the array of test function coefficients the *test vector* $\mathbf{w}$

$$\mathbf{q} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \quad (3.87)$$

With all these definitions, the weak statement, eq. (3.85) becomes

$$\mathbf{w}^T [K \mathbf{q} - \mathbf{Q}] = 0. \quad (3.88)$$

The test functions can be chosen arbitrarily, as implied by weak statement, eq. (3.82). This means that the coefficients α and β of the test function are arbitrary. In particular, we can select $\alpha = 1, \beta = 0$ or $\alpha = 0, \beta = 1$ to find two equations

$$\begin{bmatrix} 1 & 0 \end{bmatrix} [K \mathbf{q} - \mathbf{Q}] = 0, \quad \begin{bmatrix} 0 & 1 \end{bmatrix} [K \mathbf{q} - \mathbf{Q}] = 0. \quad (3.89)$$

Combining these two equations leads to

$$I [K \mathbf{q} - \mathbf{Q}] = \mathbf{0}, \quad (3.90)$$

where I is a two by two identity matrix. Of course, the identity matrix can be dropped to yield a set of algebraic equations for the solution vector

$$K \mathbf{q} = \mathbf{Q}. \quad (3.91)$$

The solution of this equation is simply $\mathbf{q} = K^{-1} \mathbf{Q}$, or

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{SL} \begin{bmatrix} 1 & L \\ L & 4L^2/3 \end{bmatrix}^{-1} p_0 L^2 \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}. \quad (3.92)$$

which yields the solution vector $a = p_0 L/S$, $b = -p_0/(2S)$. Introducing these coefficients into the assumed solution, eq. (3.83) leads to

$$U_1 = \frac{p_0 L}{S} x_1 - \frac{p_0}{2S} x_1^2 = \frac{p_0 L^2}{S} \left[\left(\frac{x_1}{L} \right) - \frac{1}{2} \left(\frac{x_1}{L} \right)^2 \right]. \quad (3.93)$$

The solution is identical to that found with the strong equilibrium statement, eq. (3.35). However, the solution process was sharply different for the two approaches. The strong equilibrium statement leads to differential equations that must be solved to obtain the solution. On the other hand, the solution of the problem based on the weak statement involves the evaluation of integrals over the beam span and the solution of a set of linear, algebraic equations.

Example 2: Tapered beam under centrifugal load

A helicopter blade of length L is rotating at an angular velocity Ω about the $\mathbf{i}_2$ axis, as depicted in fig. 3.12. The solution of this problem presented in section 3.4.8 was based on the strong statement of equilibrium. The solution of the same problem will now be derived with the help of the weak statement of equilibrium. For the case at hand, it writes

$$\int_0^L \frac{dw}{dx_1} EA(x_1) \frac{dU_1}{dx_1} dx_1 - \int_0^L w \rho A(x_1) \Omega^2 x_1 dx_1 = 0. \quad (3.94)$$

Introducing the nondimensional span variable then leads to

$$\frac{EA_0}{L} \int_0^1 w' \left(1 - \frac{\eta}{2} \right) U_1' d\eta - \rho A_0 \Omega^2 L^2 \int_0^1 w \left(\eta - \frac{\eta^2}{2} \right) d\eta = 0.$$

The solution $U_1(x_1)$ and test functions $w(x_1)$ are assumed to be of the following form

$$U_1(\eta) = a\eta + b\eta^2; \quad w(\eta) = \alpha\eta + \beta\eta^2. \quad (3.95)$$

With these approximations, the weak statement becomes

$$\frac{EA_0}{L} \int_0^1 (\alpha + 2\beta\eta)(1 - \frac{\eta}{2})(a + 2b\eta) \, dx_1 - \rho A_0 \Omega^2 L^2 \int_0^1 (\alpha\eta + \beta\eta^2)(\eta - \frac{\eta^2}{2}) \, dx_1 = 0. \quad (3.96)$$

This expression is then expanded and integrated, then the results are recast into a matrix form to find

$$K = \frac{EA_0}{L} \begin{bmatrix} 3/4 & 2/3 \\ 2/3 & 5/6 \end{bmatrix}, \quad \mathbf{Q} = \rho A_0 \Omega^2 L^2 \begin{bmatrix} 5/24 \\ 3/20 \end{bmatrix}.$$

With these definitions, the weak statement is again of the form of eq. (3.88). Following the same steps as those detailed in the previous example, the weak statement then leads to a set of linear equations, eq.(3.91). The displacement vector is now

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{L}{EA_0} \begin{bmatrix} 3/4 & 2/3 \\ 2/3 & 5/6 \end{bmatrix}^{-1} \rho A_0 \Omega^2 L^2 \begin{bmatrix} 5/24 \\ 3/20 \end{bmatrix}. \quad (3.97)$$

which yields the solution vector $a = 53\rho\Omega^2 L^3/(130E)$, $b = -19\rho\Omega^2 L^3/(130E)$. Introducing these coefficients into the assumed solution, eq. (3.95) leads to

$$U_1 = \frac{\rho\Omega^2 L^3}{E} \left(\frac{53}{130} \eta - \frac{19}{130} \eta^2 \right). \quad (3.98)$$

This solution is clearly different from that obtained from the strong statement of equilibrium, eq. (3.36). Indeed, eq. (3.36) is an *exact solution* of the problem, whereas eq. (3.98) is an *approximate solution*. The approximation was introduced by assuming the form of the solution and test function in eq. (3.95). Clearly, the assumed polynomial form of the solution cannot possibly represent the exact solution of the problem, eq. (3.36), which involves transcendental functions. Furthermore, the weak statement, eq. (3.82), requires the vanishing of an integral for all arbitrary choices of the test function. Again, the assumed polynomial form of the test function cannot possibly represent all arbitrary choices of test functions. Within the framework of this approximation, the solution process determines the values of the unknown coefficients a and b that provide the “best overall match” with the exact solution. Detailed mathematical analyses of the solution procedure [1, 2] can be used to prove that under certain restrictions, the approximate solution does converge to the exact solution as the number of unknown coefficients increases.

Although only two unknown coefficients were used here, it is interesting to note that the approximate solution is close agreement with the exact solution, as shown in fig. 3.32 which depicts the distribution of nondimensional axial displacement over the span of the beam. Table 3.1 compares the predictions of the two approaches at various locations along the span of the blade.

Finally, the axial forces in the blade are obtained by introducing eq. (3.98) into eq. (3.17) to find

$$F_1 = \frac{\rho A_0 \Omega^2 L^2}{260} (106 - 129\eta + 38\eta^2). \quad (3.99)$$

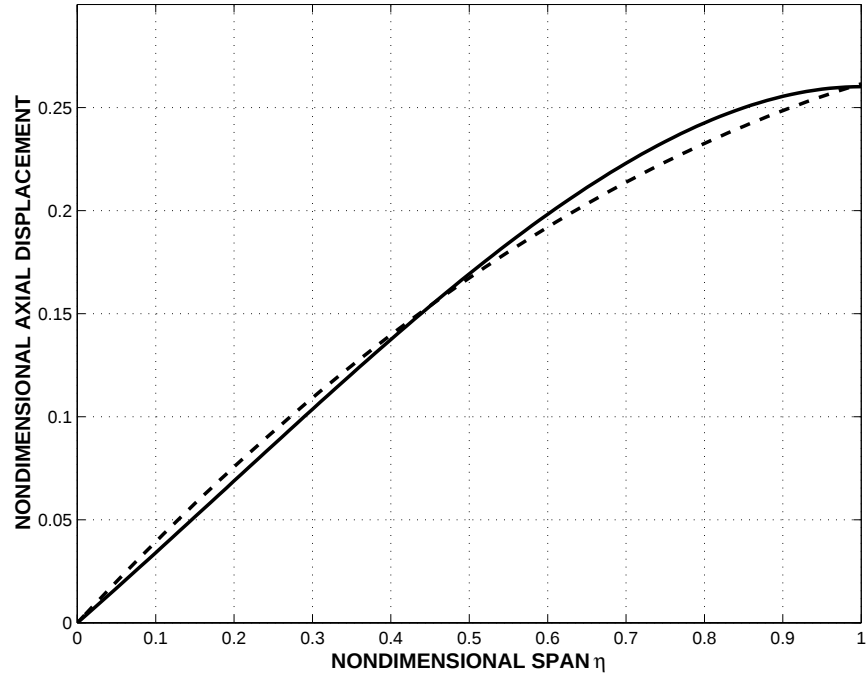


Figure 3.32: Nondimensional axial displacement $EU_1/(\rho\Omega^2 L^3)$ versus nondimensional span. The solid and dashed lines correspond to the solutions from the strong and weak equilibrium statements, respectively.

	$EU_1/(\rho\Omega^2 L^3)$ $\eta = 0.8$	$EU_1/(\rho\Omega^2 L^3)$ $\eta = 1.0$	$F_1/(\rho A_0 \Omega^2 L^2)$ $\eta = 0.5$	$F_1/(\rho A_0 \Omega^2 L^2)$ $\eta = 0.8$
Strong Statement	0.2426	0.2601	0.2292	0.0987
Weak Statement	0.2326	0.2615	0.1962	0.1043
difference	-4.1%	0.54%	-14%	5.6%

Table 3.1: Comparison of the exact and approximate solutions.

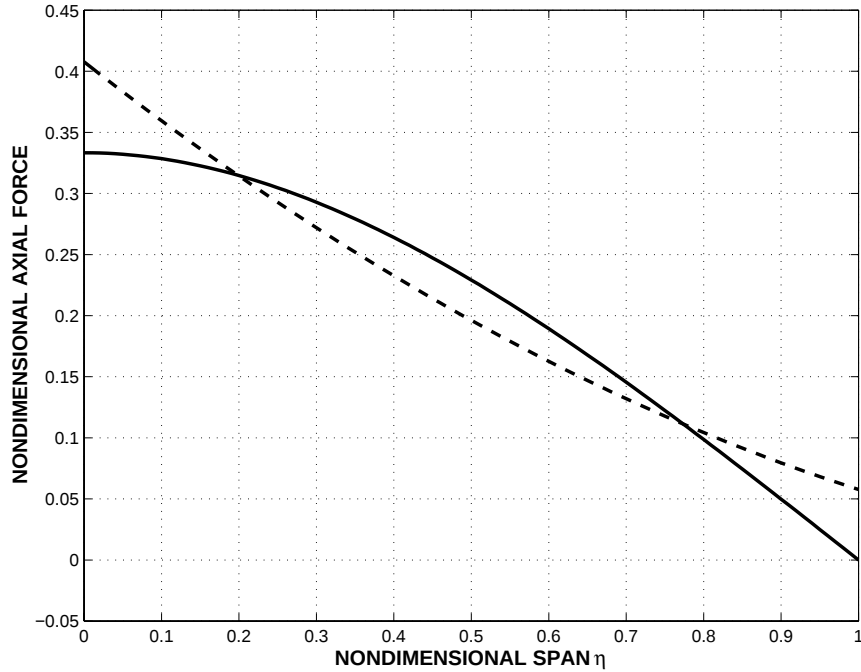


Figure 3.33: Nondimensional axial force $F_1/(A_0\rho\Omega^2L^2)$ versus nondimensional span. The solid and dashed lines correspond to the solutions from the strong and weak equilibrium statements, respectively.

Fig. 3.33 compares the axial forces predicted by the two approaches. This result is of course different from that obtained from the strong statement, eq. (3.37). However, the two solutions are in reasonable agreement as shown in fig. 3.33 that depicts the distribution of nondimensional axial force along the span of the blade. Table 3.1 lists the predictions of the two approaches at various locations along the span of the blade. It is interesting to note that the correlation in predicted displacements is clearly better than for predicted axial forces.

3.6.3 Problems

Problem 3.12

A helicopter blade of length L is rotating at an angular velocity Ω about the $\mathbf{i}_2$ axis, see fig 3.12. The blade is homogeneous and its cross-section linearly tapers from an area A_0 at the root to A_1 at the tip, $A(x_1) = A_0 + (A_1 - A_0)x_1/L$. Select $A_0 = 2 A_1$.

1. Solve the governing differential equations of this problem to find the axial displacement $U_1(x_1)$ and the axial load $F_1(x_1)$.
2. Find an approximate solution of the problem using a weak formulation. Select the following forms for the displacement field $U_1(x_1) = a x_1 + b x_1^2$ and test function $w(x_1) = \alpha x_1 + \beta x_1^2$.
3. Compare the solution obtained in parts (1) and (2). Plot the exact and approximate axial displacement fields $U_1(x_1)$ on the same plot. Plot the exact and approximate axial force $F_1(x_1)$ on the same plot.

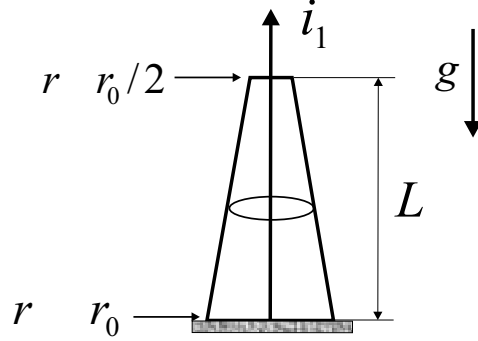


Figure 3.34: Tapered beam subjected to gravity loads

4. Comment on your results. How would you improve the approximate solution?

Hint: A mass M rotating about an axis i_2 at an angular velocity Ω is subjected to a centrifugal force $F_c = M\Omega^2 r$, where r is the distance between the mass and the axis of rotation. Hence, the helicopter blade is subjected to an axial load per unit span $p_1(x_1) = \rho A(x_1)\Omega^2 x_1$, where ρ is the material density.

Problem 3.13

Consider the tapered beam of circular cross-section subjected to gravity loads, as shown in fig. 3.34. The root section is of radius r_0 , whereas the tip section has a radius $r_0/2$. The radius linearly tapers along the length of the beam, $r(x_1) = r_0[1 - x_1/(2L)]$. The acceleration of gravity is g , the material Young's modulus E and density ρ .

1. Solve the governing differential equations of this problem to find the axial displacement $U(x_1)$ and the axial load $F_1(x_1)$.
2. Find an approximate solution of the problem using a weak formulation. Select the following forms for the displacement field $U_1(x_1) = a x_1 + b x_1^2$ and test function $w(x_1) = \alpha x_1 + \beta x_1^2$.
3. Compare the solution obtained in parts (1) and (2). Plot the exact and approximate axial displacement fields $U_1(x_1)$ on the same plot. Plot the exact and approximate axial force $F_1(x_1)$ on the same plot.

3.6.4 The weak form for beams under transverse loads

The differential equation of equilibrium of a beam under transverse loads, eq. (3.47), was derived earlier. This equation holds at all points over the span of the beam, *i.e.* for $x_1 \in [0, L]$. At the loaded end of the beam, equilibrium requires the shear force to equal the applied transverse load, $F_2(L) = P_2$, and the bending moment must vanish, $M_3^c = 0$. These two equilibrium requirements are repeated here

$$\begin{aligned} \frac{d^2 M_3^c}{dx_1^2} &= p_2, & x_1 &\in [0, L]; \\ F_2 &= P_2, & M_3^c &= 0, & x_1 &= L. \end{aligned} \quad (3.100)$$

These equations are known as the *strong statement of equilibrium* and are used to construct the following integral statement

$$\int_0^L w(x_1) \left[\frac{d^2 M_3^c}{dx_1^2} - p_2 \right] dx_1 + [w(L) (F_2 - P_2)]_L + \left[\frac{dw}{dx_1} M_3^c \right]_L = 0. \quad (3.101)$$

Note that if the beam is in equilibrium eqs. (3.100) hold, and eq. (3.101) is satisfied *for all arbitrary functions* $w(x_1)$. Next, an integration by parts is performed on the first term appearing in the integral

$$\int_0^L w(x_1) \frac{d^2 M_3^c}{dx_1^2} dx_1 = - \int_0^L \frac{dw}{dx_1} \frac{dM_3^c}{dx_1} dx_1 + \left[w \frac{dM_3^c}{dx_1} \right]_0^L.$$

The integration by parts is repeated for the first term on the right hand side of the equation and eq. (3.46) is introduced in the boundary term to yield

$$\int_0^L w(x_1) \frac{d^2 M_3^c}{dx_1^2} dx_1 = \int_0^L \frac{d^2 w}{dx_1^2} M_3^c dx_1 - \left[\frac{dw}{dx_1} M_3^c \right]_0^L - [w F_2]_0^L.$$

Introducing this result into eq. (3.101) leads to

$$\int_0^L \frac{d^2 w}{dx_1^2} M_3^c dx_1 - \int_0^L w p_2 dx_1 - w(L) P_2 - \frac{dw}{dx_1}(0) M_3^c(0) - w(0) F_2(0) = 0. \quad (3.102)$$

Since $w(x_1)$ is an entirely arbitrary function, $w(0) = dw/dx_1(0) = 0$ can be selected and this choice yields

$$\int_0^L \frac{d^2 w}{dx_1^2} M_3^c dx_1 - \int_0^L w p_2 dx_1 - w(L) P_2 = 0, \quad (3.103)$$

for all arbitrary $w(x_1)$ such that $w(0) = dw/dx_1(0) = 0$.

This integral is known as the *weak statement of equilibrium*. It is now clear that the strong statement, eq. (3.100), implies the weak statement, eq. (3.103). On the other hand, it is easily shown that the weak statement implies the strong statement. Indeed, the weak statement implies eq. (3.102), which in turns implies eq. (3.101) by reversing the integration by parts processes. Finally, the strong statement of equilibrium is implied by eq. (3.101) because this equation must hold for all arbitrary choices of $w(x_1)$.

In summary, the strong statement, eq. (3.100), and the weak statement, eq. (3.103), are two entirely equivalent statements that both express the equilibrium conditions for the beam under transverse loads.

The above reasoning still holds if different boundary conditions were applied to the problem. The test function, or its derivative, should vanish at those locations where the transverse displacement, or rotation, of the beam are prescribed, respectively. More generally, the integral appearing in the weak statement should vanish for all $w(x_1)$ that satisfy the *geometric boundary conditions*.

Example 1: Simply supported beam under a uniform load

Consider a simply supported, uniform beam of length L subjected to a uniform transverse loading $p_2(x_1) = p_0$, as depicted in fig. 3.20. The solution of this problem presented in section 3.5.9 was based on the strong statement of equilibrium. Note that this equation of equilibrium is not sufficient to lead to the solution of the problem; it must be complemented with a constitutive law and a strain displacement relationship. The solution of the same problem will now be derived with the help of the weak statement of equilibrium, eq. (3.103). Here again, this statement must be complemented by a constitutive law, eq. (3.44), and a strain displacement relationship, eq. (3.7) to yield

$$\int_0^L \frac{d^2 w}{dx_1^2} I_{33}^c \frac{d^2 U_2}{dx_1^2} dx_1 - \int_0^L w p_0 dx_1 = 0,$$

This integral must vanish for all arbitrary $w(x_1)$ such that $w(0) = w(L) = 0$. The following forms will be assumed for the solution $U_2(x_1)$ and the test functions $w(x_1)$

$$U_2(x_1) = a_1 \sin \frac{\pi x_1}{L} + a_3 \sin \frac{3\pi x_1}{L}; \quad w(x_1) = \alpha_1 \sin \frac{\pi x_1}{L} + \alpha_3 \sin \frac{3\pi x_1}{L}, \quad (3.104)$$

where a and b are two unknown coefficients, and the coefficients α and β are arbitrary since $w(x_1)$ is itself an arbitrary function. Note that for this choice, the solution satisfies the geometric boundary conditions $U_2(0) = U_2(L) = 0$, and the test functions satisfy the conditions $w(0) = w(L) = 0$. With these approximations, the weak statement becomes

$$\begin{aligned} \int_0^L I_{33}^c \left[-\alpha_1 \left(\frac{\pi}{L} \right)^2 \sin \frac{\pi x_1}{L} - \alpha_3 \left(\frac{3\pi}{L} \right)^2 \sin \frac{3\pi x_1}{L} \right] \left[-a_1 \left(\frac{\pi}{L} \right)^2 \sin \frac{\pi x_1}{L} - a_3 \left(\frac{3\pi}{L} \right)^2 \sin \frac{3\pi x_1}{L} \right] dx_1 \\ - \int_0^L p_0 \left[\alpha_1 \sin \frac{\pi x_1}{L} + \alpha_3 \sin \frac{3\pi x_1}{L} \right] dx_1 = 0. \end{aligned}$$

This expression is now expanded and integrated, then the results are recast into a matrix form as

$$K = \frac{\pi^4 I_{33}^c}{2L^3} \begin{bmatrix} 1 & 0 \\ 0 & 81 \end{bmatrix}, \quad \mathbf{Q} = \frac{2p_0 L}{\pi} \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}.$$

With these definitions, the weak statement is again of the form of eq. (3.88), and the weak statement then implies a set of linear equations, eq.(3.91). The displacement vector is now

$$\begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \frac{2L^3}{\pi^4 I_{33}^c} \begin{bmatrix} 1 & 0 \\ 0 & 81 \end{bmatrix}^{-1} \frac{2p_0 L}{\pi} \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}.$$

which yields the solution vector $a_1 = 4p_0 L^4 / (\pi^5 I_{33}^c)$, $a_3 = 4p_0 L^4 / (243\pi^5 I_{33}^c)$. Introducing these coefficients into the assumed solution, eq. (3.95) leads to

$$U_2 = \frac{4p_0 L^4}{\pi^5 I_{33}^c} \left[\sin \frac{\pi x_1}{L} + \frac{1}{243} \sin \frac{3\pi x_1}{L} \right]. \quad (3.105)$$

This solution is clearly different from that obtained from the strong statement of equilibrium, eq. (3.68). Indeed, eq. (3.68) is an *exact solution* of the problem, whereas eq. (3.105) is an *approximate solution*. The approximation was introduced by assuming the form of the solution and test function in eq. (3.104).

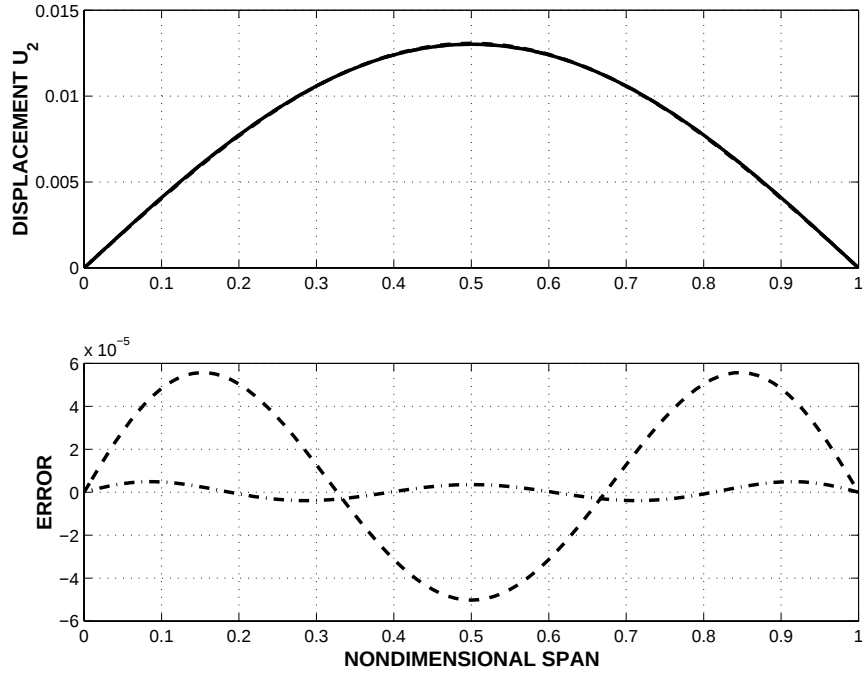


Figure 3.35: Transverse displacement U_2 (top figure) and discrepancy between approximate and exact solutions (bottom figure) versus nondimensional span. Exact solution: solid line; Approximate solution: dashed line (*case 1*), dash-dotted line (*case 2*).

Fig. 3.35 depicts the distribution of nondimensional transverse displacement $I_{33}^c U_2 / (p_0 L^4)$ over the span of the beam for both exact and approximate solutions. Two approximate solutions are shown: *case 1* only includes the term in $\sin \pi x_1 / L$, whereas *case 2* represents the complete expression in eq. (3.105). At the scale of the figure, the exact and approximate solutions are indistinguishable. The lower portion of the figure shows the discrepancy between the exact solution and both approximate solutions. The excellent agreement between the various solutions is also apparent in table 3.2.

Finally, the bending moment distribution is obtained by introducing eq. (3.105) into

	$I_{33}^c U_2 / (p_0 L^4)$ $\eta = 0.25$	$I_{33}^c U_2 / (p_0 L^4)$ $\eta = 0.5$	$M_3^c / (p_0 L^2)$ $\eta = 0.25$	$M_3^c / (p_0 L^2)$ $\eta = 0.5$
Strong Statement	9.277×10^{-03}	13.02×10^{-03}	0.09375	0.125
Weak Statement	9.243×10^{-03}	13.07×10^{-03}	0.09122	0.129
<i>case 1</i>	(-0.4%)	(0.4%)	(-2.7%)	(3.2%)
Weak Statement	9.281×10^{-03}	13.02×10^{-03}	0.0946	0.124
<i>case 2</i>	(0.04%)	(-0.03%)	(0.9%)	(-0.8%)

Table 3.2: Comparison of the exact and approximate solutions.

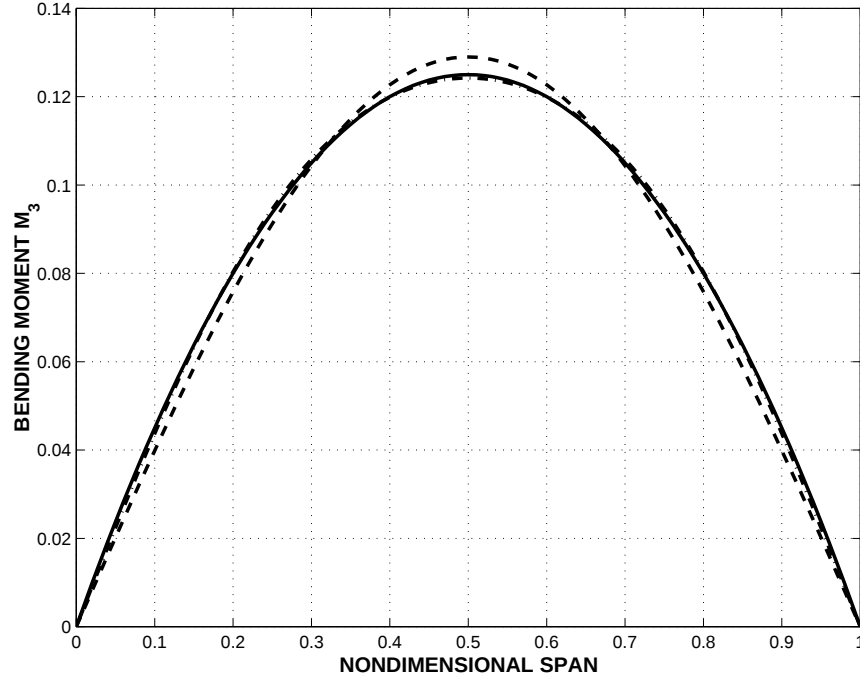


Figure 3.36: Bending moment M_3^c versus nondimensional span. Exact solution: solid line; Approximate solution: dashed line (*case 1*), dash-dotted line (*case 2*).

eq. (3.44) to find

$$M_3^c = \frac{4p_0L^2}{\pi^3} \left[\sin \frac{\pi x_1}{L} + \frac{1}{27} \sin \frac{3\pi x_1}{L} \right]. \quad (3.106)$$

Fig. 3.36 compares the bending moments predicted by the two approaches. Here again, excellent agreement is observed between the various solutions, as confirmed by table 3.2.

3.6.5 Problems

Problem 3.14

Consider a simply supported, uniform beam of length L subjected to a uniform transverse loading $p_2(x_1) = p_0$, as depicted in fig. 3.20.

1. Solve the governing differential equations of this problem to find the transverse displacement $U_2(x_1)$, the bending moment $M_3^c(x_1)$, and the shear force $F_2(x_1)$.
2. Find an approximate solution of the problem using a weak formulation. Select the following forms for the displacement field $U_2(x_1) = \sum_{i=1}^{N_{\max}} a_i \sin(2i-1)\pi x_1/L$ and test function $w(x_1) = \sum_{i=1}^{N_{\max}} \alpha_i \sin(2i-1)\pi x_1/L$.
3. Plot the exact and approximate transverse displacement fields $U_2(x_1)$ on the same plot. For the approximate solutions use $N_{\max} = 1, 2, 3, 4$, and 5.
4. Plot the exact and approximate bending moments $M_3^c(x_1)$ on the same plot.

5. Plot the exact and approximate shear forces $F_2(x_1)$ on the same plot.

3.6.6 The Principle of Virtual Work

The weak statement of equilibrium was introduced in the above section. This statement involves arbitrary test or weighting functions $w(x_1)$ that satisfy the geometric boundary conditions, and is ideally suited for the derivation of approximate solutions of beam problems, as demonstrated in the examples. The weak statement was introduced through a purely mathematical process, and consequently, its physical interpretation is not clear at this point. To help cast the weak statement in a more physical framework, the test functions are interpreted as follows

$$w(x_1) = \delta U_1(x_1), \quad (3.107)$$

where the right hand side read: “*arbitrary variation in displacements*” or “*virtual displacements*”. Note that eq. (3.107) is purely a matter of notation: the test functions are arbitrary functions that satisfies the geometric boundary conditions, and the virtual displacements are arbitrary displacements that satisfy the geometric boundary conditions. The two notations are entirely equivalent, although the latter carries a more physical meaning. Introducing this new notation into the weak statement of equilibrium, eq. (3.82), yields

$$\int_0^L \delta \left(\frac{dU_1}{dx_1} \right) F_1 dx_1 = \int_0^L \delta U_1(x_1) p_1 dx_1 + \delta U_1(L) P_1, \quad (3.108)$$

for all arbitrary $\delta U_1(x_1)$ that satisfy the geometric boundary conditions.

The work done by a force equals the product of this force by the displacement of its point of application in the direction of the force. Hence, the work done by the tip load P_1 is $P_1 U_1(L)$. It is then logical to interpret the term $\delta U_1(L) P_1$ as the *virtual work* performed by the tip load P_1 . Similarly, the work done by the distributed load p_1 is $\int_0^L U_1(x_1) p_1 dx_1$, and $\int_0^L \delta U_1(x_1) p_1 dx_1$ is then the virtual work done by this distributed load. In summary, the right hand side of eq. (3.108) can be interpreted as the virtual work done by the externally applied loads. The left hand side of this equation corresponds to the the virtual work done by the axial force F_1 . Indeed $\int_0^L F_1 dU_1 = \int_0^L dU_1/dx_1 F_1 dx_1$ is the work done by the axial force, and $\int_0^L \delta(dU_1/dx_1) F_1 dx_1$ is then the virtual work expression.

Eq. (3.108) can thus be interpreted as follows

Principle 1 (Principle of Virtual Work) *A beam is in equilibrium if the virtual work done by the internal forces equals the virtual work done by the externally applied loads, for all arbitrary virtual displacements that satisfy the geometric boundary conditions.*

This statement is known as the *Principle of Virtual Work*. It is clear that the principle of virtual work is entirely equivalent to the weak statement of equilibrium, which in turns, was shown in section 3.6.1 to be entirely equivalent to the equilibrium conditions for the beam. Hence, the principle of virtual work is simply a statement of equilibrium of the beam, and to solve practical problems, it must be complemented with a constitutive law and a strain displacement relationship.

A more physical interpretation of the weak statement of equilibrium for beams under transverse loads, eq. (3.103), is found by introducing virtual transverse displacements $\delta U_2(x_1)$. This leads to

$$\int_0^L \delta \left(\frac{d^2 U_2}{dx_1^2} \right) M_3^c dx_1 = \int_0^L \delta U_2(x_1) p_2 dx_1 + \delta U_2(L) P_2, \quad (3.109)$$

for all arbitrary $\delta U_2(x_1)$ that satisfy the geometric boundary conditions.

The right hand side of this expression clearly represents the virtual work done by the applied distributed load p_2 and tip concentrated load P_2 . The work done by the bending moment is $\int_0^L M_3^c d(dU_2/dx_1) = \int_0^L d^2 U_2/dx_1^2 M_3^c dx_1$. Hence, the left hand side of the expression can be interpreted as the virtual work done by the bending moment. As a result, although the expressions of the principle of virtual work for beams under axial loads, eq. (3.108), and transverse loads, eq. (3.109), are different, their physical interpretation is identical, Principle 1.

Approximate solutions for beam problems can be obtained from the principle of virtual work by following a procedure identical to that used with the weak statement of equilibrium. Consider, for instance, the first example of section 3.6.4. The principle of virtual work writes

$$\int_0^L \delta \left(\frac{dU_1}{dx_1} \right) S \frac{dU_1}{dx_1} dx_1 = \int_0^L \delta U_1(x_1) p_0 dx_1.$$

The following forms are then assumed for the solution $U_1(x_1)$ and the virtual displacements $\delta U_1(x_1)$

$$U_1(x_1) = a x_1 + b x_1^2; \quad \delta U_1(x_1) = \delta \alpha x_1 + \delta \beta x_1^2, \quad (3.110)$$

where a and b are two unknown coefficients, and $\delta \alpha$ and $\delta \beta$ are arbitrary coefficients since the virtual displacement is itself arbitrary. These expressions should be compared with eq. (3.83). The rest of the procedure is identical to that outlined in section 3.6.4, and identical results are obtained.

3.6.7 The Principle of Minimum Total Potential Energy

The principle of virtual work developed in the above section is solely a statement of equilibrium of the beam and must be complemented with a constitutive law and a strain displacement relationship to solve practical problems. In fact, the statement of the principle, eqs. (3.108) or (3.109), does not involve the stiffness characteristics of the structure, nor the strains. It is often convenient to incorporate all the information required for the solution of a problem into one single statement. To this effect, the constitutive law, eq. (3.17), and the strain displacement relationship, eq. (3.7), are introduced into the principle of virtual work, eq. (3.108) to find

$$\int_0^L \delta \left(\frac{dU_1}{dx_1} \right) S \frac{dU_1}{dx_1} dx_1 = \int_0^L \delta U_1(x_1) p_1 dx_1 + \delta U_1(L) P_1. \quad (3.111)$$

The integral on the left hand side can be expressed as follows

$$\int_0^L \delta \left(\frac{dU_1}{dx_1} \right) S \frac{dU_1}{dx_1} dx_1 = \int_0^L \delta \left[\frac{1}{2} S \left(\frac{dU_1}{dx_1} \right)^2 \right] dx_1 = \delta \int_0^L \frac{1}{2} S \epsilon_1^2 dx_1.$$

$1/2 S\epsilon_1^2$ is the strain energy density function, $a(\epsilon_1)$, defined in eq. (3.32). As discussed in section 3.4.7, if a differential element of the beam is deformed by an axial strain ϵ_1 , the elastic energy stored in the slice of the beam is $1/2 S\epsilon_1^2$. The total amount of elastic energy stored in the deformed beam is then $A(\epsilon_1) = \int_0^L a(\epsilon_1) dx_1$. Hence,

$$\int_0^L \delta\left(\frac{dU_1}{dx_1}\right) S \frac{dU_1}{dx_1} dx_1 = \delta \int_0^L a(\epsilon_1) dx_1 = \delta A(\epsilon_1).$$

Eq. (3.111) now becomes

$$\delta A(\epsilon_1) = \int_0^L \delta U_1(x_1) p_1 dx_1 + \delta U_1(L) P_1. \quad (3.112)$$

This statement is interpreted as follows

Principle 2 *A beam is in equilibrium if virtual changes in the total strain energy equal the virtual work done by the externally applied loads, for all arbitrary virtual displacements that satisfy the geometric boundary conditions.*

An even more compact statement of this principle can be obtained if the externally applied loads are assumed to be derived from a potential, that is

$$p_1(x_1) = -\frac{\partial \phi}{\partial U_1}; \quad P_1 = -\frac{\partial \Psi}{\partial U_1}, \quad (3.113)$$

where ϕ is the *potential of the distributed loads*, and Ψ the *potential of the concentrated load*. The term on right hand side of eq. (3.112) now becomes

$$\begin{aligned} \int_0^L \delta U_1(x_1) p_1 dx_1 + \delta U_1(L) P_1 &= - \int_0^L \delta U_1(x_1) \frac{\partial \phi}{\partial U_1} dx_1 - \delta U_1(L) \frac{\partial \Psi}{\partial U_1} \\ &= -\delta \left[\int_0^L \phi dx_1 + \Psi \right] = -\delta \Phi, \end{aligned}$$

where Φ is the *total potential of the externally applied loads*. Introducing this result in eq. (3.112) yields

$$\delta A(\epsilon_1) = -\delta \Phi.$$

Finally, the *total potential energy* Π of the system is defined

$$\Pi = A + \Phi, \quad (3.114)$$

and clearly

$$\delta \Pi = 0. \quad (3.115)$$

This statement is interpreted as follows

Principle 3 (Principle of Minimum Total Potential Energy) *A beam is in equilibrium if the total potential energy is a minimum with respect to all arbitrary choices of virtual displacements that satisfy the geometric boundary conditions.*

For beam subjected transverse loads, the total potential energy is still given by eq. (3.114). The strain energy is due to curvature, and the total strain energy in the beam is given by eq. (3.66). The total potential of the applied load combines the potentials of the distributed and concentrated transverse loads,

$$p_2(x_1) = -\frac{\partial \phi}{\partial U_2}; \quad P_2 = -\frac{\partial \Psi}{\partial U_2}, \quad (3.116)$$

respectively. The statement of the principle of minimum total potential energy, Principle 3, remains unchanged.

The three principles derived thus far all express the equilibrium conditions for the beam, since all were derived from the weak statement of equilibrium. However, these principles are not all equivalent. The weak statement of equilibrium and the principle of virtual work are entirely equivalent and solely express equilibrium conditions for the beam; they provide no information about either constitutive law or strain displacement relationship. Equilibrium conditions stem from applying Newton's Law to structures: the sum of all the forces acting on a differential element of the structure must vanish. Since Newton's Law always applies to any structure, so do the weak statement of equilibrium and the principle of virtual work.

On the other hand, the principle of minimum total potential energy is also a statement of equilibrium, but both constitutive law and strain displacement relationship have been worked into the statement of the principle. The principle of minimum total potential energy summarizes in a single, concise statement all the information required for the solution of structural problems. However, it is important to note that assumptions were made in the derivation of this principle. First, the existence of a strain energy density function was assumed. For linear elastic material behavior these function are given by eqs. (3.32) and (3.65), for beam under axial and transverse loads, respectively. Strain energy density functions exist for more general material behavior such as nonlinear elastic or hyper-elastic materials. However, they do not always exist: for instance if a beam is bent past the elastic limit for the material, plastic deformation occur. In this plastic regime, no strain energy density function exists. Similarly, visco-elastic and visco-plastic material behavior do not give rise to strain energy density functions.

The second assumption made in the derivation of the principle of minimum total potential energy is that the applied load are derived from a potential. Many types of loading can indeed be derived from a potential. For instance, the potentials corresponding to a tip transverse load or a distributed load are $\Psi = -P_2 U_2(L)$, and $\phi = -p_2(x_1) U_2(x_1)$, respectively. Gravity loads are also derived from a potential. If the gravity vector $\mathbf{g} = g \mathbf{i}_2$, the potential of the gravity loads is $\phi = m(x_1) g U_2(x_1)$, where $m(x_1)$ is the mass per unit span of the beam. The resulting distributed gravity loading is $p_2 = -m(x_1) g$, as expected. On the other hand, some types of loading of importance to aerospace structures, such as aerodynamic loads acting on a wing, cannot be derived from a potential.

In summary, the weak statement of equilibrium and the principle of virtual work are the most general principles as they apply to all structural problem. Constitutive laws and strain displacement relationships must complement these principles for the solution of specific problems. Applications of the principle of minimum total potential energy are limited to problem involving materials with constitutive laws that give rise to a strain energy density function, and applied loads that are derived from a potential. Finally, principle 2 is usefull

when a strain energy density function exist, but applied loads cannot be derived from a potential.

3.6.8 The Principle of Virtual Work: general solution procedure

The examples presented in the previous sections have demonstrated that approximate solutions of complex structural problem can be derived from the principle of virtual work. In this section, the solution procedure is described in general terms to underline the salient features of the method. The first step of the solution procedure is to assume specific forms for the displacements and virtual displacements

$$U_1(x_1) = \sum_{i=1}^N h_i(x_1) q_i; \quad \delta U_1(x_1) = \sum_{i=1}^N g_i(x_1) w_i, \quad (3.117)$$

where the q_i are unknown coefficients, often called *degrees of freedom*, that determine the solution of the problem, and the w_i a set of arbitrary coefficients reflecting the arbitrary nature of the virtual displacements. The functions $h_i(x_1)$ and $g_i(x_1)$ are sets of arbitrary functions called *shape functions* that each satisfy the geometric boundary conditions. In the examples treated in the previous sections, monomials or trigonometric functions have been selected as shape functions. Polynomial or transcendental functions can also be used so long as they form a set of linearly independent functions that satisfy the geometric boundary conditions. It is customary to select the same shape functions for both displacements and virtual displacement, *i.e.* $h_i(x_1) = g_i(x_1)$. Although this is a convenient choice, it is not required by the principle of virtual work. Expression (3.117) represent an approximation to the problem: the solution is assumed to consists of the linear combination of a finite number of shape functions, the magnitude of each coefficient q_i indicates how much the corresponding shape function contributes to the final solution. For the exact solution, $U_1(x_1)$ must be determined at each $x_1 \in [0, L]$, *i.e.* at an infinite number of points; the solution is said to present an infinite number of degrees of freedom. Clearly, assumption (3.117) reduces the number of degrees of freedom from infinity to N .

The same remarks can be made about the assumed form of the virtual displacements. The principle of virtual work requires that the virtual work done by the internal forces be equal to that done by the externally applied forces for all arbitrary choices of the virtual displacements that satisfy the geometric boundary conditions. This clearly calls for virtual displacements involving an infinite number of degrees of freedom. Here again, assumption (3.117) reduces the number of choices for the virtual displacements from infinity to N .

It is convenient to recast the expressions for the assumed displacements and virtual displacements, eq. (3.117), to a matrix form

$$U_1(x_1) = H(x_1) \mathbf{q}; \quad \delta U_1(x_1) = H(x_1) \mathbf{w}, \quad (3.118)$$

where $\mathbf{q}^T = [q_1, q_2, \dots, q_N]$ is the N dimensional solution array that stores the N degrees of freedom of the problem, and $\mathbf{w}^T = [w_1, w_2, \dots, w_N]$ the array of arbitrary coefficients representing the virtual displacements. The *displacement interpolation matrix*, $H(x_1)$, is defined as

$$H(x_1) = [h_1(x_1), h_2(x_1), h_3(x_1), \dots, h_{N(x_1)}]. \quad (3.119)$$

As is customary, the same shape functions were chosen to be identical for both displacements and virtual displacement, *i.e.* $h_i(x_1) = g_i(x_1)$, leading to identical interpolation matrices for both quantities.

Next, the assumed displacements are introduced in the strain displacement relationship, eq. (3.7), to find the axial strain distribution

$$\epsilon_1(x_1) = B(x_1) \mathbf{q}, \quad (3.120)$$

where the *strain interpolation matrix* is

$$B(x_1) = [h'_1(x_1), h'_2(x_1), h'_3(x_1), \dots, h'_N(x_1)]. \quad (3.121)$$

It is clear that virtual strains are written in a similar manner

$$\delta\left(\frac{dU_1}{dx_1}\right) = B(x_1) \mathbf{w}. \quad (3.122)$$

With these notations, the principle of virtual work becomes

$$\int_0^L \mathbf{w}^T B^T S B \mathbf{q} \, dx_1 = \int_0^L \mathbf{w}^T H^T p_1 \, dx_1 + \mathbf{w}^T H^T(L) P_1. \quad (3.123)$$

Clearly, the array of coefficients $\mathbf{q}$ and $\mathbf{w}$ can be extracted from the integrals, to yield

$$\mathbf{w}^T \left[\int_0^L B^T S B \, dx_1 \right] \mathbf{q} = \mathbf{w}^T \left[\int_0^L H^T p_1 \, dx_1 \right] + \mathbf{w}^T H^T(L) P_1. \quad (3.124)$$

The stiffness matrix is defined as

$$K = \int_0^L B(x_1)^T S(x_1) B(x_1) \, dx_1. \quad (3.125)$$

Note that for a given choice of the shape function the stiffness matrix is an array of coefficients that can be obtained by integration along the beam span. Each entry of the matrix can be viewed as an average of the axial stiffness distribution weighted by products the assumed axial strain distributions. Next, the load vector is introduced

$$\mathbf{Q} = \int_0^L H^T(x_1) p_1(x_1) \, dx_1 + H^T(L) P_1. \quad (3.126)$$

Here again, for a given choice of the shape function, the load vector is an array of coefficients that can be obtained by integration along the beam span. Each entry of the load vector corresponds to an average of the applied load distribution weighted by the various shape functions. This leads to the following compact expression of the principle

$$\mathbf{w}^T [K \mathbf{q} - \mathbf{Q}] = 0. \quad (3.127)$$

Since the array $\mathbf{w}$ can be chosen arbitrarily, the following N choices are selected: $\mathbf{w}^T = [1, 0, 0, \dots, 0]$, $\mathbf{w}^T = [0, 1, 0, \dots, 0]$, $\mathbf{w}^T = [0, 0, 1, \dots, 0]$, and finally $\mathbf{w}^T = [0, 0, 0, \dots, 1]$.

Each new choice give rise to a new equation, and the combination of this collection of N equations writes

$$I_N [K \mathbf{q} - \mathbf{Q}] = \mathbf{0}, \quad (3.128)$$

where I_N indicates the $N \times N$ identity matrix. Of course, this identity matrix can be dropped to yield a set of algebraic equations for the solution vector

$$K \mathbf{q} = \mathbf{Q}, \quad (3.129)$$

the solution of which simply is $\mathbf{q} = K^{-1} \mathbf{Q}$. Once the degrees of freedom stored in the solution vector $\mathbf{q}$ are found, the solution of the problem, eq. (3.117), is entirely determined. The strains then follow from eq. (3.120), and finally, the constitutive law yields the internal forces.

The general solution procedure using the principle of virtual work can be summarized by the following steps

1. Select a set of shapes functions $h_i(x_1)$ that satisfy the geometric boundary conditions.
2. Construct the displacement interpolation matrix, eq. (3.119), and strain interpolation matrix, eq. (3.121).
3. Compute the $N \times N$ stiffness matrix K according to eq. (3.125) and load vector $\mathbf{Q}$ from eq. (3.126).
4. Solve the $N \times N$ set of linear equations $K \mathbf{q} = \mathbf{Q}$ for the solution vector $\mathbf{q}$.
5. Determine the strain distribution from eq. (3.120), and the internal forces from the constitutive law, eq. (3.17).

From a mathematical stand point, this procedure involves the following types of operation: integrations over the beam span for evaluating the stiffness matrix and load vector, the solution of a set of linear algebraic equations to obtain the solution vector, and linear algebra operations for the determination of the strain and internal force distributions. Of course, the process become increasingly tedious as the number of degrees of freedom increases. However, all the required operations are easily performed on a computer using numerical integration procedures for the evaluation of the stiffness matrix and load vector, and standard linear algebra software packages for all remaining operations.

The general method was presented for beams under axial loads, but a nearly identical process can be developed for beams subjected to transverse loads. At first, specific forms for the transverse displacements and virtual displacements are assumed

$$U_2(x_1) = \sum_{i=1}^N h_i(x_1) q_i; \quad \delta U_2(x_1) = \sum_{i=1}^N h_i(x_1) w_i. \quad (3.130)$$

The displacement field is then $U_2(x_1) = H(x_1) \mathbf{q}$, where the displacement interpolation matrix is here again given by eq. (3.119). Next, the curvatures are computed from eq. (3.7) to find $\kappa_3 = B(x_1) \mathbf{q}$ where the strain interpolation matrix is

$$B(x_1) = [h_1''(x_1), h_2''(x_1), h_3''(x_1), \dots, h_N'']. \quad (3.131)$$

The rest of the procedure then mirrors that presented above with the stiffness matrix and load vector defined as

$$K = \int_0^L B(x_1)^T I_{33}^c(x_1) B(x_1) dx_1, \quad (3.132)$$

and

$$\mathbf{Q} = \int_0^L H^T(x_1) p_2(x_1) dx_1 + H^T(L) P_2, \quad (3.133)$$

respectively.

It is interesting to note that the entire solution process is automated once the shape functions have been selected. For instance, example 1 of section 3.6.1 is an axial load problem and is characterized by a displacement interpolation matrix $H(x_1) = [x_1, x_1^2]$ and corresponding strain interpolation matrix $B(x_1) = [1, 2x_1]$. On the other hand, example 1 of section 3.6.4 is a transverse load problem with corresponding quantities $H(x_1) = [\sin \pi x_1/L, \sin 3\pi x_1/L]$ and $B(x_1) = -(\pi/L)^2 [\sin \pi x_1/L, 9 \sin 3\pi x_1/L]$.

3.6.9 The Principle of Minimum Total Potential Energy: general solution procedure

The principle of minimum total potential energy also leads to a general procedure for obtaining approximate solutions of structural problem. The methodology closely mirrors that used in conjunction with the principle of virtual work. The first step of the solution procedure is to assume a specific form for the displacements field

$$U_1(x_1) = \sum_{i=1}^N h_i(x_1) q_i; \quad (3.134)$$

Next, the strain are expressed in terms of the assumed displacements, resulting in eq. (3.120), with the strain interpolation matrix defined in eq. (3.121). The total strain energy in the beam is now

$$A(\epsilon_1) = \frac{1}{2} \int_0^L S \epsilon_1^2 dx_1 = \frac{1}{2} \mathbf{q}^T \left[\int_0^L B^T S B dx_1 \right] \mathbf{q} = \frac{1}{2} \mathbf{q}^T K \mathbf{q}, \quad (3.135)$$

where the stiffness matrix K is identical to that defined in eq. (3.125). Next, the total potential of the externally applied loads becomes

$$\Phi = \int_0^L \phi dx_1 + \Psi = -\mathbf{q}^T \left[\int_0^L H^T(x_1) p_1(x_1) dx_1 \right] - \mathbf{q}^T H^T(L) P_1 = -\mathbf{q}^T \mathbf{Q}. \quad (3.136)$$

Here again, the load vector $\mathbf{Q}$ is equal to that found earlier, eq. (3.126). The total potential energy of the system, eq. (3.114), now simply writes

$$\Pi(\mathbf{q}) = \frac{1}{2} \mathbf{q}^T K \mathbf{q} - \mathbf{q}^T \mathbf{Q}. \quad (3.137)$$

According to the principle of minimum total potential energy, Principle 3, this expression should be a minimum with respect to all possible choice of the displacements. The possible

choices of displacements are now limited to the choice of the degrees of freedom $\mathbf{q}$. In other words, the total potential energy is now a sole function of $\mathbf{q}$, *i.e.* $\Pi = \Pi(\mathbf{q})$. The minimum of a function of several variables is found by setting its derivatives equal to zero. Hence, the condition for minimization of Π is

$$\frac{\partial \Pi}{\partial \mathbf{q}} = K\mathbf{q} - \mathbf{Q} = \mathbf{0}. \quad (3.138)$$

The solution vector $\mathbf{q}$ must satisfy a set of linear algebraic equations, identical to that found based on the principle of virtual work, eq. (3.129).

The general solution procedure using the principle of minimum total potential energy can be summarized by the following steps

1. Select a set of shapes functions $h_i(x_1)$ that satisfy the geometric boundary conditions.
2. Construct the displacement interpolation matrix, eq. (3.119), and strain interpolation matrix, eq. (3.121).
3. Compute the total strain energy of the system $A(\mathbf{q}) = 1/2 \mathbf{q}^T K \mathbf{q}$, where the stiffness matrix is given by eq. (3.125).
4. Compute the total potential of the externally applied loads $\Phi(\mathbf{q}) = -\mathbf{q}^T \mathbf{Q}$, where the load vector $\mathbf{Q}$ is given by eq. (3.126).
5. Solve the set of linear equations $K \mathbf{q} = \mathbf{Q}$ for the solution vector $\mathbf{q}$.
6. Determine the strain distribution from eq. (3.120), and the internal forces from the constitutive law, eq. (3.17).

The solution procedures based on the principle of virtual work and principle of minimum total potential energy are obviously closely related. Although the physical interpretation of the intermediate quantities is different, the major elements of the procedures, namely the stiffness matrix and load vector, are identical, and so are the final solutions.

The general method presented here for beams under axial loads, can be readily applied for beams subjected to transverse loads, by using the appropriate expressions for the stiffness matrix, eq. (3.132), and load vector, eq. (3.133).

When the structural system to be analyzed comprises several elastic components, such as beams and springs, the strain energies of the various elastic elements are simply added together to find the total strain energy. Consider, for instance a beam with a concentrated spring at location $x_1 = \alpha L$ subjected to transverse loads. The system now consists of the beam and spring, and the total strain energy is

$$A = \frac{1}{2} \int_0^L I_{33}^c \left(\frac{d^2 U_2}{dx_1^2} \right)^2 dx_1 + \frac{1}{2} k U_2^2(\alpha L). \quad (3.139)$$

The stiffness matrix is now $K = K_b + K_s$, where K_b is associated with the strain energy of the beam in bending, and K_s with the strain energy stored in the spring. K_b is given by eq. (3.132), and the strain energy in the spring gives rise to K_s

$$\frac{1}{2} k U_2^2(\alpha L) = \frac{1}{2} \mathbf{q}^T [H^T(\alpha L) k H(\alpha L)] \mathbf{q} = \frac{1}{2} \mathbf{q}^T K_s \mathbf{q}. \quad (3.140)$$

Example 1: Simply supported beam with two elastic spring supports

Fig. 3.23 depicts a simply supported beam of span L supported by two spring of stiffness k located at stations $x_1 = \alpha L$ and $(1 - \alpha)L$, and subjected to a uniform transverse loading p_0 . This problem was treated as example 3 of section 3.5.9, where the exact solution of the problem was obtained. This problem will now be analyzed with the principle of minimum total potential energy. The system under consideration consists of an elastic beam and two elastic spring. The strain energy for beam in bending is given by eq. (3.66), whereas the strain energy for the springs is

$$A_s = \frac{1}{2} k U_2^2(\alpha L) + \frac{1}{2} k U_2^2((1 - \alpha)L),$$

where k is the spring stiffness constant. The strain energy for the entire system is then the sum to the strain energies for the various components of the system

$$A = \frac{1}{2} \int_0^L I_{33}^c \left(\frac{d^2 U_2}{dx_1^2} \right)^2 dx_1 + \frac{1}{2} k U_2^2(\alpha L) + \frac{1}{2} k U_2^2((1 - \alpha)L).$$

The displacement interpolation matrix is now selected as

$$H = [\sin \pi \eta, \sin 3\pi \eta, \sin 5\pi \eta],$$

where η is the nondimensional span variable. Note that the shape functions satisfy the geometric boundary conditions $h_i(0) = h_i(1) = 0$. The corresponding strain interpolation matrix becomes

$$B = -\pi^2 [\sin \pi \eta, 3^2 \sin 3\pi \eta, 5^2 \sin 5\pi \eta].$$

The stiffness matrix for the beam then become

$$K_b = \frac{I_{33}^c}{L^3} \int_0^1 B(\eta)^T B(\eta) d\eta = \frac{I_{33}^c \pi^4}{2L^3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 5^4 \end{bmatrix}.$$

The stiffness matrix associated with the spring is

$$\begin{aligned} K_s &= H^T(\alpha) k H(\alpha) + H^T(1 - \alpha) k H(1 - \alpha) \\ &= 2k \begin{bmatrix} \sin^2 \pi \alpha & \sin \pi \alpha \sin 3\pi \alpha & \sin \pi \alpha \sin 5\pi \alpha \\ \sin 3\pi \alpha \sin \pi \alpha & \sin^2 3\pi \alpha & \sin 3\pi \alpha \sin 5\pi \alpha \\ \sin 5\pi \alpha \sin \pi \alpha & \sin 5\pi \alpha \sin 3\pi \alpha & \sin^2 5\pi \alpha \end{bmatrix}. \end{aligned}$$

The stiffness matrix for the entire structure is now

$$K = K_b + K_s = \frac{I_{33}^c}{L^3} \begin{bmatrix} \frac{\pi^4}{2} + 2k^* \sin^2 \pi \alpha & 2k^* \sin \pi \alpha \sin 3\pi \alpha & 2k^* \sin \pi \alpha \sin 5\pi \alpha \\ 2k^* \sin 3\pi \alpha \sin \pi \alpha & \frac{3^4 \pi^4}{2} + 2k^* \sin^2 3\pi \alpha & 2k^* \sin 3\pi \alpha \sin 5\pi \alpha \\ 2k^* \sin 5\pi \alpha \sin \pi \alpha & 2k^* \sin 5\pi \alpha \sin 3\pi \alpha & \frac{5^4 \pi^4}{2} + 2k^* \sin^2 5\pi \alpha \end{bmatrix},$$

where $k^* = kL^3/I_{33}^c$ is the nondimensional spring stiffness. Next, the load vector associated with the uniform transverse load is computed

$$\mathbf{Q} = p_0 L \int_0^1 H(\eta) \, d\eta = p_0 L \frac{2}{\pi} \begin{bmatrix} 1 \\ 1/3 \\ 1/5 \end{bmatrix}$$

The solution of the problem is then obtained by solving the linear set of equations $K \mathbf{q} = \mathbf{Q}$ to yield the degrees of freedom $\mathbf{q}$. This step is most easily performed numerically, since the inversion of the 3×3 stiffness matrix is a rather arduous task. If a single shape function had been selected, *i.e.* if $H = [\sin \pi \eta]$, the solution would simply write

$$q_1 = \frac{p_0 L^4}{I_{33}^c} \frac{2}{\pi(\pi^4/2 + 2k^* \sin^2 \pi \alpha)}.$$

The exact solution of the problem, eq. (3.74), will now be compared with the above approximate solution. Three cases, denoted *case 1*, *2*, and *3*, will be examined, corresponding to the displacement interpolation matrices $H = [\sin \pi \eta]$, $H = [\sin \pi \eta, \sin 3\pi \eta]$, and $H = [\sin \pi \eta, \sin 3\pi \eta, \sin 5\pi \eta]$, respectively. Fig. 3.37 shows the nondimensional transverse displacement $I_{33}^c U_2 / (p_0 L^4)$ for the exact and approximate solutions with the following choice of the parameters: $\alpha = 0.3$ and $k^* = 10^4$. Due to the symmetry of the problem, the solution is presented over a half span only. Excellent correlation is observed between the exact and approximate solutions; at the scale of the figure, the predictions for *cases 2* and *3* are on top of the exact solution. Table. 3.3 quantifies the observed errors for the various approximations.

The exact distribution of bending moment is given by eq. (3.75), and fig. 3.38 depicts the nondimensional bending moment $M_3^c / (p_0 L^2)$ for the various solutions. Note that the exact solution presents a cusp at $\eta = 0.3$. This feature is not reproduced by the approximate solutions that consist of a sum of smooth functions. However, as the number of degrees of freedom increases, the quality of the approximation improves, as detailed in table 3.3. The errors in bending moments are much larger than those observed for the transverse displacements.

Finally, fig. 3.39 shows the nondimensional shear force $F_2 / (p_0 L)$ for the exact solution, eq. (3.76), and the approximate solutions. The exact shear force distribution presents a discontinuity at $\eta = 0.3$ corresponding to the concentrated force the spring applies to the beam. Here again, this feature cannot be reproduced by the approximate solutions that consist of a sum of smooth functions. As the number of degrees of freedom increases, the approximate solution converge to shear force value corresponding to the average of the exact shear forces to the left and right of the discontinuity. However, this convergence is quite slow, as detailed in table 3.3. The larger errors observed in the predictions of bending moments and shear forces are expected since these quantities are obtained from higher order derivatives of the displacements.

Example 2: Simply supported beam on an elastic foundation

Consider a simply supported beam of length L subjected to a uniform transverse load p_0 and supported by an elastic foundation of distributed stiffness k , as depicted in fig. 3.24. This

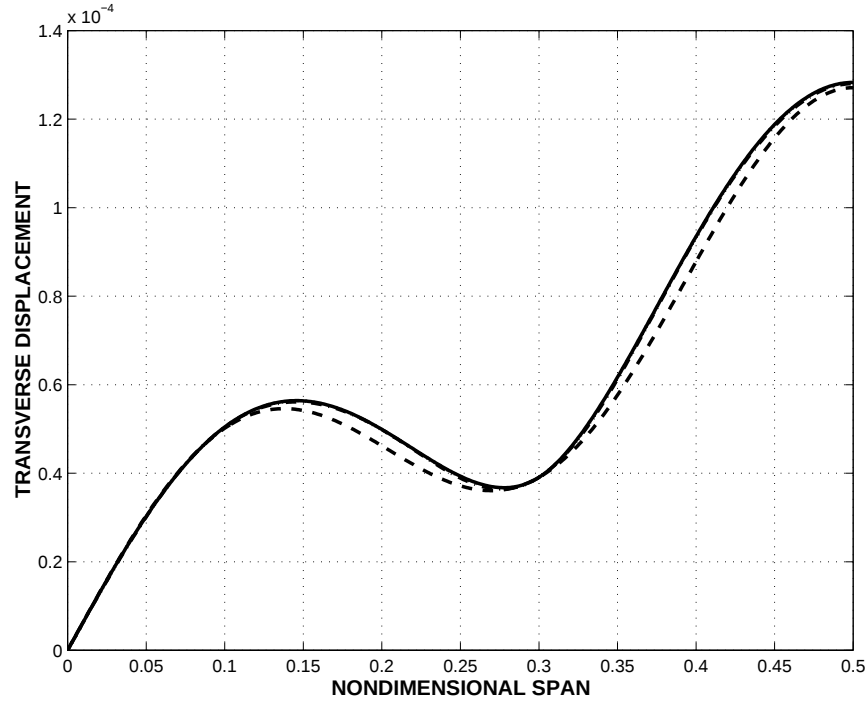


Figure 3.37: Transverse displacement $I_{33}^c U_2 / (p_0 L^4)$ for the exact and approximate solutions versus nondimensional span. Exact solution: solid line; approximate solution *case 1*: dashed line, *case 2*: dash-dotted line, *case 3*: dotted line.

	Exact Solution	<i>Case 1</i> Error	<i>Case 2</i> Error	<i>Case 3</i> Error
Transverse displacement [10^{-05}]				
$\eta = 0.15$	5.6351	3.9%	-0.50%	-0.050%
$\eta = 0.30$	3.9068	0.03%	0.003%	0.0001%
$\eta = 0.45$	11.882	-2.6%	-0.3%	-0.08%
Bending moment [10^{-03}]				
$\eta = 0.15$	-5.1476	-1.9%	-7.5%	3.6%
$\eta = 0.30$	12.205	-42.%	-19.%	-14.%
$\eta = 0.45$	-6.5452	6.1%	-3.6%	2.4%
Shearing force [10^{-02}]				
$\eta = 0.15$	-4.0683	64.%	-4.9%	5.7%
$\eta = 0.30$	0.9320	-160.%	14.%	35.%
$\eta = 0.45$	5.000	95.%	6.9%	9.75%

Table 3.3: Comparison of the exact and approximate solutions.

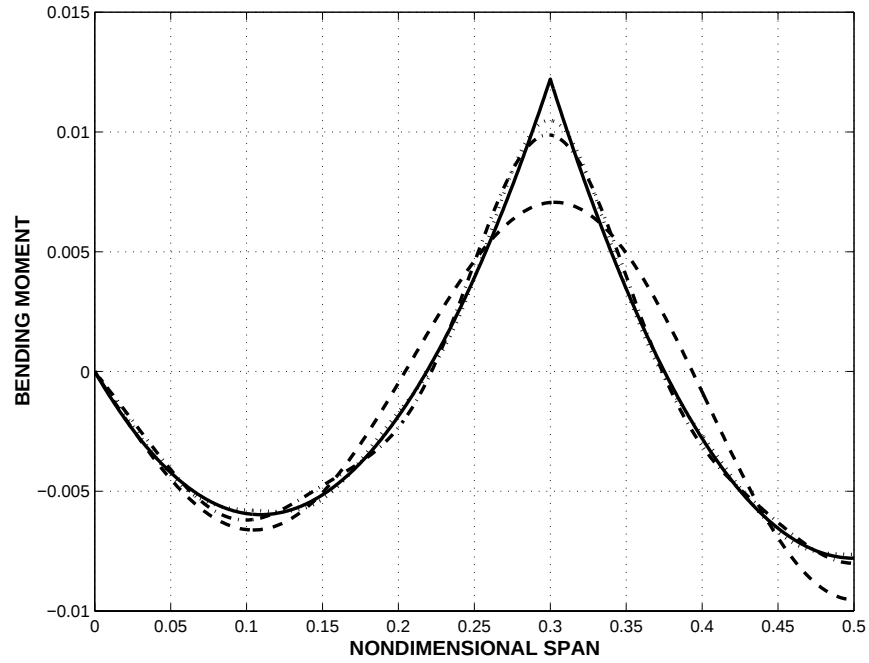


Figure 3.38: Bending moment $M_3^c/(p_0 L^2)$ for the exact and approximate solutions versus nondimensional span. Exact solution: solid line; approximate solution *case 1*: dashed line, *case 2*: dash-dotted line, *case 3*: dotted line.

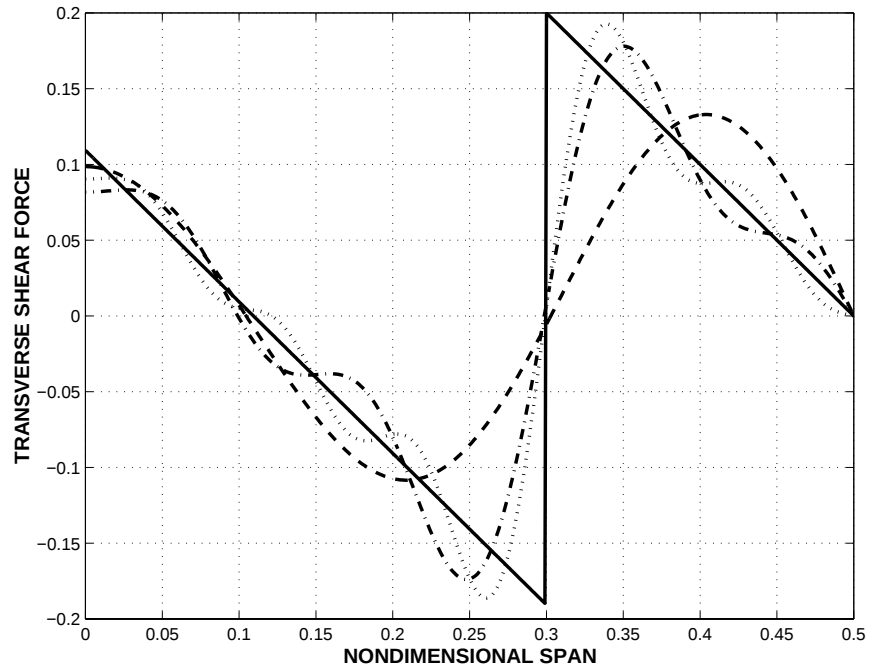


Figure 3.39: Shear force $F_2/(p_0 L)$ for the exact and approximate solutions versus nondimensional span. Exact solution: solid line; approximate solution *case 1*: dashed line, *case 2*: dash-dotted line, *case 3*: dotted line.

problem was treated as example 4 of section 3.5.9, where the exact solution of the problem was obtained. The strain energy for beam in bending is given by eq. (3.66), whereas the strain energy for the elastic foundation is

$$A_{ef} = \frac{1}{2} \int_0^L k U_2^2(x_1) \, dx_1.$$

The strain energy for the entire system is then

$$A = \frac{1}{2} \int_0^L I_{33}^c \left(\frac{d^2 U_2}{dx_1^2} \right)^2 \, dx_1 + \frac{1}{2} \int_0^L k U_2^2(x_1) \, dx_1.$$

The displacement interpolation matrix is now selected as

$$H = [\sin \pi \eta, \sin 3\pi \eta, \sin 5\pi \eta],$$

where η is the nondimensional span variable. Note that the shape functions satisfy the geometric boundary conditions $h_i(0) = h_i(1) = 0$. The corresponding strain interpolation matrix becomes

$$B = -\pi^2 [\sin \pi \eta, 3^2 \sin 3\pi \eta, 5^2 \sin 5\pi \eta].$$

The stiffness matrix for the beam then become

$$K_b = \frac{I_{33}^c}{L^3} \int_0^1 B(\eta)^T B(\eta) \, d\eta = \frac{I_{33}^c \pi^4}{2L^3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 5^4 \end{bmatrix}.$$

The stiffness matrix associated with the elastic foundation is

$$K_{ef} = kL \int_0^1 H^T(\eta) H(\eta) \, d\eta = \frac{kL}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The stiffness matrix for the entire structure is now

$$K = K_b + K_{ef} = \frac{I_{33}^c}{L^3} \frac{1}{2} \begin{bmatrix} \pi^4 + k^* & 0 & 0 \\ 0 & 3^4 \pi^4 + k^* & 0 \\ 0 & 0 & 5^4 \pi^4 + k^* \end{bmatrix},$$

where $k^* = kL^4/I_{33}^c$ is the nondimensional distributed stiffness. Next, the load vector associated with the uniform transverse load is computed

$$\mathbf{Q} = p_0 L \int_0^1 H(\eta) \, d\eta = p_0 L \frac{2}{\pi} \begin{bmatrix} 1 \\ 1/3 \\ 1/5 \end{bmatrix}$$

The solution of the problem is then obtained by solving the linear set of equations $K \mathbf{q} = \mathbf{Q}$ to yield the degrees of freedom $\mathbf{q}$. The solution for the transverse displacement writes

$$\frac{I_{33}^c U_2}{p_0 L^4} = \frac{4}{\pi} \left[\frac{\sin \pi \eta}{\pi^4 + k^*} + \frac{\sin 3\pi \eta}{3(3^4 \pi^4 + k^*)} + \frac{\sin 5\pi \eta}{5(5^4 \pi^4 + k^*)} \right].$$

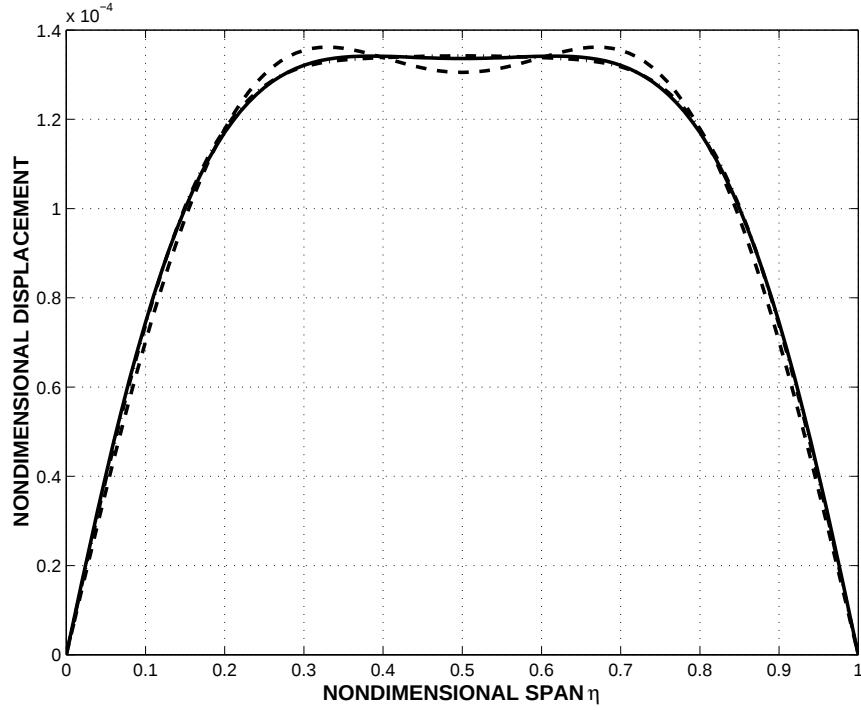


Figure 3.40: Transverse displacement $I_{33}^c U_2 / (p_0 L^4)$ for the exact and approximate solutions versus nondimensional span. Exact solution: solid line; approximate solution *case 1*: dashed line, *case 2*: dash-dotted line.

The exact solution of the problem, eq. (3.77), will now be compared with the above approximate solution. Two cases, denoted *case 1* and *case 2*, will be examined, corresponding to the displacement interpolation matrices $H = [\sin \pi \eta, \sin 3\pi \eta]$, and $H = [\sin \pi \eta, \sin 3\pi \eta, \sin 5\pi \eta]$, respectively. Fig. 3.40 shows the nondimensional transverse displacement $I_{33}^c U_2 / (p_0 L^4)$ for the exact and approximate solutions when $k^* = 8 \times 10^3$. Good correlation is observed and quantitative results are shown in Table. 3.4.

The exact distribution of bending moment is given by eq. (3.78), and fig. 3.41 depicts the nondimensional bending moment $M_3^c / (p_0 L^2)$ for the various solutions. Here again, the errors in bending moments are much larger than those observed for the transverse displacements. However, as the number of degrees of freedom increases, the quality of the approximation improves, as shown in table 3.4.

3.6.10 Problems

Problem 3.15

A simply supported beam of span L is supported by two spring of stiffness k located at stations $x_1 = \alpha L$ and $(1 - \alpha)L$, and is subjected to a uniform transverse loading p_0 , as depicted in fig. 3.23. Use the Principle of Minimum Total Potential Energy to find an approximate solution for this problem using the following assumed transverse displacement field $U_2 = \sum_{n=1}^{n_{max}} a_n \sin(2n-1)\pi x_1 / L$.

	Exact Solution	<i>Case 1</i> Error	<i>Case 2</i> Error
Transverse displacement [10^{-04}] $\eta = 0.50$	1.3364	-2.3%	0.4%
Bending moment [10^{-03}] $\eta = 0.10$	-3.5381	-32.%	-6.4%

Table 3.4: Comparison of the exact and approximate solutions.

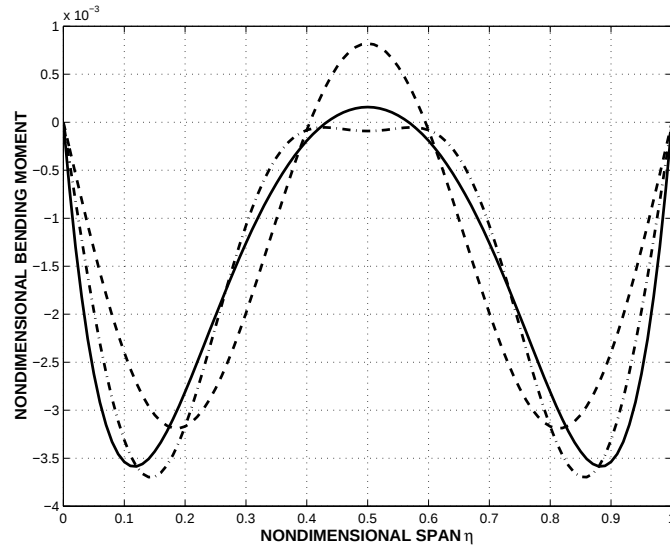


Figure 3.41: Bending moment $M_3^c/(p_0 L^2)$ for the exact and approximate solutions versus nondimensional span. Exact solution: solid line; approximate solution *case 1*: dashed line, *case 2*: dash-dotted line.

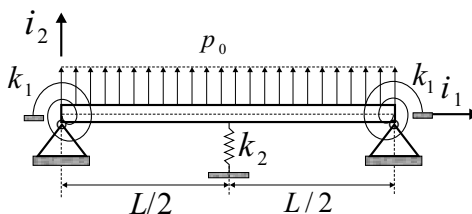


Figure 3.42: Simply supported beam with mid-span and end point springs.

1. Find the exact solution of the problem from the solution of the governing differential equation and associated boundary conditions.
2. Find the approximate displacements distribution for the problem. On a single graph, plot the exact solution and the approximate solutions using an increasing number of terms in the series, *i.e.* plots with $n_{max} = 1, 2, 3, 4, 5$. Quantify the error in maximum displacement as n_{max} increases. Use $k^* = kL^3/I_{33}^c = 10^4$ and $\alpha = 0.3$.
3. Find the approximate bending moment distribution for the problem. On a single graph, plot the exact solution and the approximate solutions using an increasing number of terms in the series, *i.e.* plots with $n_{max} = 1, 2, 3, 4, 5$. Quantify the error in maximum bending moment as n_{max} increases.
4. Check the overall equilibrium of the problem for both the exact and approximate solutions. In view of the symmetry of the problem, overall equilibrium implies $p_0 L = 2R + 2kU_2(L/3)$, where R is the reaction at either end of the beam, and $kU_2(L/3)$ the force either elastic spring. Comment on your results.
5. Based on a simple free body diagram, show that for the exact solution the shear force presents a discontinuity at the elastic springs. What happens in your approximate solution? Comment and explain your results. On a single graph, plot the exact solution and the approximate shear force distribution using an increasing number of terms in the series, *i.e.* plots with $n_{max} = 1, 2, 3, 4, 5$. Quantify the error in maximum bending moment as n_{max} increases.

Problem 3.16

Consider a simply supported, uniform beam of length L with two end point torsional springs of stiffness k_1 and a mid-span spring of stiffness k_2 . The beam, shown in fig. 3.42, is subjected to a uniform transverse loading $p_2(x_1) = p_0$.

1. Solve the governing differential equations of this problem to find the transverse displacement $U_2(x_1)$, the bending moment $M_3^c(x_1)$, and the shear force $F_2(x_1)$.
2. Find an approximate solution of the problem using the principle of minimum total potential energy. Select the following forms for the displacement field $U_2(x_1) = a_1 \sin \pi x_1/L + a_3 \sin 3\pi x_1/L$.
3. Plot the exact and approximate transverse displacement fields $U_2(x_1)$ on the same plot.
4. Plot the exact and approximate bending moments $M_3^c(x_1)$ on the same plot.

5. Plot the exact and approximate shear forces $F_2(x_1)$ on the same plot.
6. Explain why the approximation is so poor. Hint: look at the bending moment plots.

It will be convenient to work with nondimensional spring stiffnesses $k_1^* = k_1 L / I_{33}^c$ and $k_2^* = k_2 L^3 / I_{33}^c$. For your plots, select $k_1^* = 10.0$ and $k_2^* = 100.0$.

Problem 3.17

A simply supported beam of span L is subjected to forces of magnitude P located at stations $x_1 = \alpha L$ and $(1 - \alpha)L$, as depicted in fig. 3.27. The beam has a bending stiffness I_0 and is reinforced in its central portion where its bending stiffness is I_1 . Use the Principle of Minimum Total Potential Energy to find an approximate solution for this problem using the following assumed transverse displacement field $U_2 = \sum_{n=1}^{n_{max}} a_n \sin(2n - 1)\pi x_1 / L$.

1. Find the exact solution of the problem from the solution of the governing differential equation and associated boundary conditions.
2. Find the approximate displacements distribution for the problem. On a single graph, plot the exact solution and the approximate solutions using an increasing number of terms in the series, *i.e.* plots with $n_{max} = 1, 2, 3, 4, 5$. Quantify the error in maximum displacement as n_{max} increases. Use $I_1 / I_0 = 2$ and $\alpha = 0.3$.
3. Find the approximate bending moment distribution for the problem. On a single graph, plot the exact solution and the approximate solutions using an increasing number of terms in the series, *i.e.* plots with $n_{max} = 1, 2, 3, 4, 5$. Quantify the error in maximum bending moment as n_{max} increases.
4. Based on a simple free body diagram, show that for the exact solution the shear force presents a discontinuity at the point of application of the transverse loads P . What happens in your approximate solution? Comment and explain your results. On a single graph, plot the exact solution and the approximate shear force distribution using an increasing number of terms in the series, *i.e.* plots with $n_{max} = 1, 2, 3, 4, 5$. Quantify the error in maximum bending moment as n_{max} increases.

Problem 3.18

The lower beam depicted in fig. 3.28 is of length $2L$ and is simply supported at both ends. The upper beam of length $L + a$ is cantilevered at the root, supported by the lower beam at point A , and subjected to a uniform transverse loading p_0 . Both upper and lower beams have a uniform bending stiffness I_0 .

1. Find the exact solution for the transverse deflection of the lower beam under a mid-span concentrated load. Show that the lower beam can be replaced by a concentrated spring of stiffness constant

$$k_{eq} = \frac{6I_0}{L^3}.$$

2. Find the exact solution of the problem from the solution of the governing differential equation and associated boundary conditions for the upper beam. Replace the lower beam by the equivalent spring of stiffness constant given above. Model the interaction between the upper beam and equivalent spring by a force X , yet unknown. The magnitude of this force is found by equating the displacements of the upper beam and spring at point A .

3. Find an approximate displacement distribution for the upper beam. Use the Principle of Minimum Total Potential Energy using the following assumed transverse displacement field

$$U_2(x_1) = \sum_{n=0}^{n_{\max}} a_n x_1^{2+n}.$$

Use $L/a = 2$. On a single graph, plot the exact and approximate solutions using an increasing number of terms in the series, *i.e.* plots with $n_{\max} = 0, 1, 2, 3, 4$. Quantify the error in maximum displacement as $n_{\max}$ increases.

4. Find the approximate bending moment distribution for the problem. On a single graph, plot the exact and approximate solutions using an increasing number of terms in the series, *i.e.* plots with $n_{\max} = 0, 1, 2, 3, 4$. Quantify the error in maximum bending moment as $n_{\max}$ increases.
5. Based on a simple free body diagram, show that for the exact solution the shear force presents a discontinuity at point *A*. What happens in your approximate solution? Comment and explain your results. On a single graph, plot the exact and approximate shear force distributions using an increasing number of terms in the series, *i.e.* plots with $n_{\max} = 0, 1, 2, 3, 4$. Quantify the error in maximum shear force as $n_{\max}$ increases.

Problem 3.19

The cantilevered beam depicted in fig. 3.29 is of length L , uniform bending stiffness I_{33} , and is subjected to a uniform distributed load p_0 . A concentrated spring of stiffness constant k is connected to the beam at a distance a from its root.

1. Find the exact solution of the problem from the solution of the governing differential equations and associated boundary conditions for the cantilevered beam. It will be necessary to solve the problem in two parts, for the segments to the right and left of the spring, then applying continuity conditions at this point. It will be convenient to define the non-dimensional spring constant

$$k^* = \frac{ka^3}{I_{33}}$$

2. Find an approximate displacement distribution for the beam. Use the Principle of Minimum Total Potential Energy using the following assumed transverse displacement field

$$U_2(x_1) = \sum_{n=0}^{n_{\max}} a_n x_1^{2+n}.$$

Use $L/a = 3$ and $k^* = 100$. On a single graph, plot the exact and approximate solutions using an increasing number of terms in the series, *i.e.* plots with $n_{\max} = 0, 1, 2, 3, 4$. Quantify the error in maximum displacement as $n_{\max}$ increases.

3. Find the approximate bending moment distribution for the problem. On a single graph, plot the exact and approximate solutions using an increasing number of terms in the series, *i.e.* plots with $n_{\max} = 0, 1, 2, 3, 4$. Quantify the error in maximum bending moment as $n_{\max}$ increases.

4. Based on a simple free body diagram, show that for the exact solution the shear force presents a discontinuity at the connection point for the spring. What happens in your approximate solution? Comment and explain your results. On a single graph, plot the exact and approximate shear force distributions using an increasing number of terms in the series, *i.e.* plots with $n_{\max} = 0, 1, 2, 3, 4$. Quantify the error in maximum shear force as $n_{\max}$ increases.

Chapter 4

Three-Dimensional Beam Theory

In the previous chapter Euler-Bernoulli theory was developed for beams under axial and transverse loads. However, the analysis was limited to a planar deformation of the beam. This behavior can be observed, for instance, when the cross-section of the beam presents a plane of symmetry and the applied loads are acting in this plane of symmetry.

In numerous practical applications, the cross section of the beam is of arbitrary shape and the applied loads act along several distinct directions. Consider an aircraft wing: the cross-section is of a complex shape involving curved skins and two or more spars, and the wing is subjected lift and drag forces. In the case of a helicopter blade, a very large centrifugal force generated by the rotation of the blade is also present. Similarly, machine components often operate in a complex, three-dimensional loading environment.

Fig. 4.1 shows a beam of arbitrary cross-sectional shape subjected to a complex three-dimensional loading. This loading consists of distributed and concentrated axial and transverse loads, as well as distributed and concentrated moments. The axial and transverse distributed loads $p_1(x_1)$, $p_2(x_1)$, and $p_3(x_1)$ act in the direction $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, respectively. The same convention is used for the concentrated loads P_1 , P_2 , and P_3 . The distributed moments $q_2(x_1)$ and $q_3(x_1)$ act about the axes $\mathbf{i}_2$ and $\mathbf{i}_3$, respectively. The concentrated moments Q_2 and Q_3 act about the same axes. Fig. 4.1 depicts concentrated forces and moments acting at the tip of the beam, but in practical situations, such concentrated loads could be applied at any span-wise location. The notation used in this text for the various loads is summarized in table 4.1

This three-dimensional loading is general, with an important exception: no torsional loads are applied, and the transverse loads are assumed to be applied in such a way that the beam will bend without twisting. This important restriction will be removed in a later

Loading Type	Notation	Units
Distributed loads	$p_1(x_1), p_2(x_1), p_3(x_1)$	N/m
Concentrated loads	P_1, P_2, P_3	N
Distributed moments	$q_2(x_1), q_3(x_1)$	$N.m/m$
Concentrated moments	Q_2, Q_3	$N.m$

Table 4.1: Loading components acting on the beam.

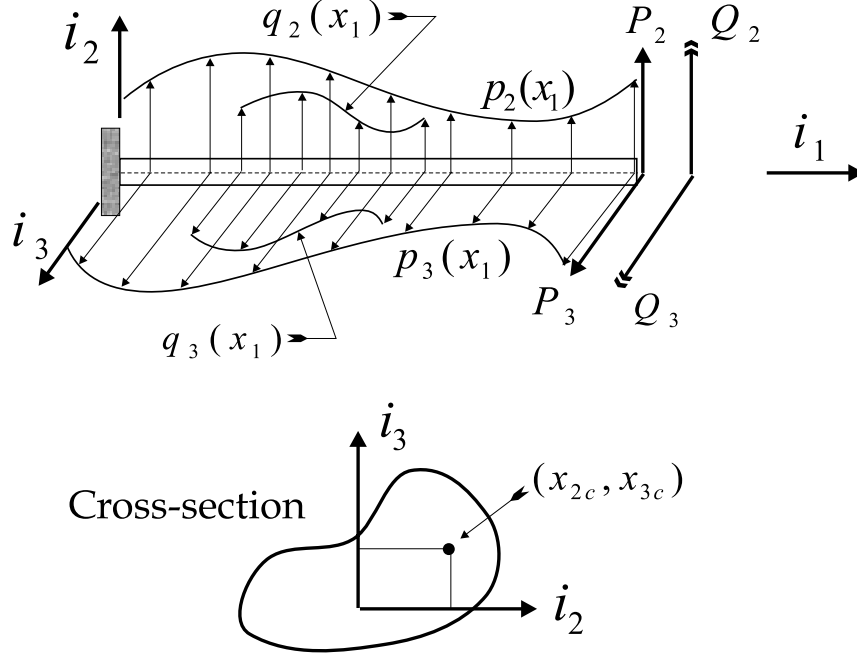


Figure 4.1: Beam with arbitrary three-dimensional loading.

chapter, after the study of the torsional behavior of beams.

As mentioned earlier, the cross-section of the beam is of arbitrary shape. The origin of the axes is at an arbitrary point of the cross-section and the orientation of the axes $\mathbf{i}_2$ and $\mathbf{i}_3$ within the plane of the section is arbitrary, as depicted in fig. 4.1. The centroid of the section of coordinates (x_{2c}, x_{3c}) is as yet undetermined.

4.1 Kinematic description

The development of the three-dimensional beam theory starts with the three Euler-Bernoulli assumptions discussed in section (3.1). These assumptions are of a purely kinematic nature and were shown to imply the following displacements field

$$u_1(x_1, x_2, x_3) = U_1(x_1) + (x_3 - x_{3c})\Phi_2(x_1) - (x_2 - x_{2c})\Phi_3(x_1); \quad (4.1)$$

$$u_2(x_1, x_2, x_3) = U_2(x_1); \quad u_3(x_1, x_2, x_3) = U_3(x_1). \quad (4.2)$$

The corresponding strain field was shown to be:

$$\varepsilon_2 = 0; \quad \varepsilon_3 = 0; \quad \gamma_{23} = 0; \quad (4.3)$$

$$\gamma_{12} = 0; \quad \gamma_{13} = 0; \quad (4.4)$$

$$\varepsilon_1(x_1, x_2, x_3) = \epsilon_1(x_1) + (x_3 - x_{3c}) \kappa_2(x_1) - (x_2 - x_{2c}) \kappa_3(x_1); \quad (4.5)$$

4.2 Sectional constitutive law

Consider now a linear elastic, isotropic material for which the stress-strain relationship is adequately described by Hooke's Law, eq. (3.15). Since the cross-section does not deform in its own plane, the stresses σ_2 and σ_3 acting in the plane of the section are far smaller than the axial stresses σ_1 and Hooke's law was shown to reduce to (3.15). The axial stress distribution is found by introducing eq. (4.5) into eq. (3.15) to find

$$\sigma_1(x_1, x_2, x_3) = E [\epsilon_1(x_1) + (x_3 - x_{3c}) \kappa_2(x_1) - (x_2 - x_{2c}) \kappa_3(x_1)] \quad (4.6)$$

The axial force F_1 , and the bending moments M_2^c and M_3^c about the centroid are now evaluated by introducing this axial stress distribution into eqs. (3.9) and (3.12), respectively, to find

$$\begin{bmatrix} F_1(x_1) \\ M_2^c(x_1) \\ M_3^c(x_1) \end{bmatrix} = \begin{bmatrix} S & S_2^c & -S_3^c \\ S_2^c & I_{22}^c & -I_{23}^c \\ -S_2^c & -I_{23}^c & I_{33}^c \end{bmatrix} \begin{bmatrix} \epsilon_1(x_1) \\ \kappa_2(x_1) \\ \kappa_3(x_1) \end{bmatrix} \quad (4.7)$$

where the following sectional stiffness coefficients were defined:

$$S = \int_{\Omega} E \, d\Omega; \quad S_2^c = \int_{\Omega} E (x_2 - x_{2c}) \, d\Omega; \quad S_3^c = \int_{\Omega} E (x_3 - x_{3c}) \, d\Omega; \quad (4.8)$$

$$I_{22}^c = \int_{\Omega} E (x_3 - x_{3c})^2 \, d\Omega; \quad I_{33}^c = \int_{\Omega} E (x_2 - x_{2c})^2 \, d\Omega; \quad (4.9)$$

$$I_{23}^c = \int_{\Omega} E (x_2 - x_{2c})(x_3 - x_{3c}) \, d\Omega. \quad (4.10)$$

The axial stiffness S was found in section (3.4) to characterize the axial stiffness of the beam and in section (3.5) the bending stiffness I_{33}^c was found to characterize the bending behavior of the beam about the axis $\mathbf{i}_3$. The bending stiffness I_{22}^c plays the same role, but for bending about the axis $\mathbf{i}_2$. I_{23} is called the *cross bending stiffness*.

Equations (4.7) express a linear relationship between the sectional stress resultants and the sectional strains. They are the constitutive laws for the cross-section of the beam. It is important to note that the axial force is not simply proportional to the axial strain, nor are the bending moments simply proportional to the curvatures. Indeed, the behavior is fully coupled through the sectional coupling stiffness coefficients S_2^c , S_3^c , and I_{23}^c . An axial force F_1 will appear as a result of an axial strain ϵ_1 , but also in the presence of curvatures κ_2 or κ_3 . Similarly, a bending moment appears as a result of either κ_2 or κ_3 curvatures, but also in the presence of an axial strain ϵ_1 .

4.3 Sectional equilibrium equations

To complete the theory, equilibrium equations must now be derived. Consider an infinitesimal slice of the beam of length dx_1 as depicted in fig. 4.2. Summing all the forces in the axial direction yields the axial equilibrium equation

$$\frac{dF_1}{dx_1} = -p_1(x_1) \quad (4.11)$$

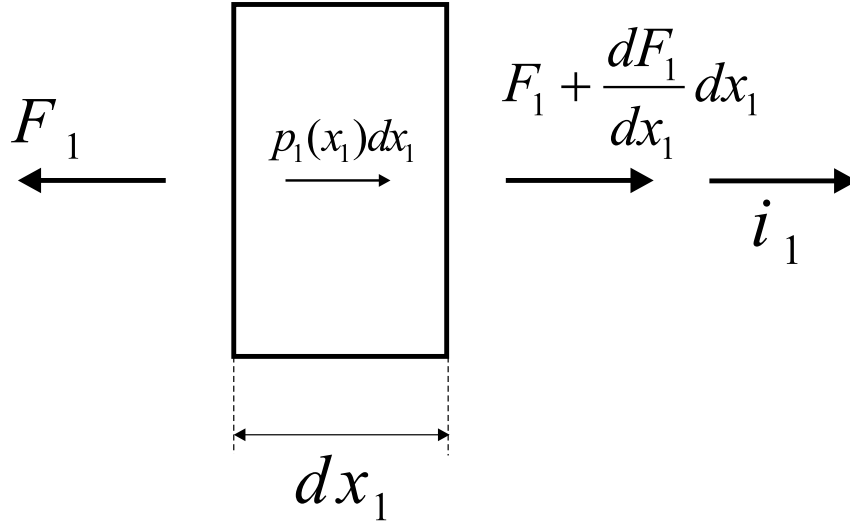


Figure 4.2: Free body diagram for the axial forces.

Fig. 4.3 depicts the transverse loads and bending moments acting on an infinitesimal slice of the beam, focusing on the $(\mathbf{i}_1, \mathbf{i}_2)$ plane. A summation of the forces along axis $\mathbf{i}_2$ gives the transverse equilibrium equation

$$\frac{dF_2}{dx_1} = -p_2(x_1). \quad (4.12)$$

Finally, a summation of the moments taken about the centroid (x_{2c}, x_{3c}) along an axis parallel to $\mathbf{i}_3$ yields

$$\frac{dM_3^c}{dx_1} + F_2 = -q_3(x_1) + p_1(x_1)(x_{2a} - x_{2c}). \quad (4.13)$$

Since the moment equilibrium is taken about the centroid, the distributed axial load p_1 applied at a point of coordinates (x_{2a}, x_{3a}) creates a distributed moment $p_1 dx_1 (x_{2a} - x_{2c})$, where $(x_{2a} - x_{2c})$ is the moment arm.

Similarly, fig. 4.3 also depicts the transverse loads and bending moments acting on an infinitesimal slice of the beam, focusing on the $(\mathbf{i}_1, \mathbf{i}_3)$ plane. Summing the forces along axis $\mathbf{i}_3$ gives the second transverse equilibrium equation

$$\frac{dF_3}{dx_1} = -p_3(x_1), \quad (4.14)$$

and the summing of the moments taken about the centroid along an axis parallel to $\mathbf{i}_2$ leads to

$$\frac{dM_2^c}{dx_1} - F_3 = -q_2(x_1) - p_1(x_1)(x_{3a} - x_{3c}). \quad (4.15)$$

The shear forces F_2 and F_3 can be eliminated from the equilibrium equations by taking a derivative of (4.15) and (4.13), then introducing eqs. (4.14) and (4.12), respectively, to yield the bending moment equilibrium equations

$$\frac{d^2 M_2^c}{dx_1^2} = -p_3(x_1) - \frac{d}{dx_1}[(x_{3a} - x_{3c})p_1(x_1) + q_2(x_1)]; \quad (4.16)$$

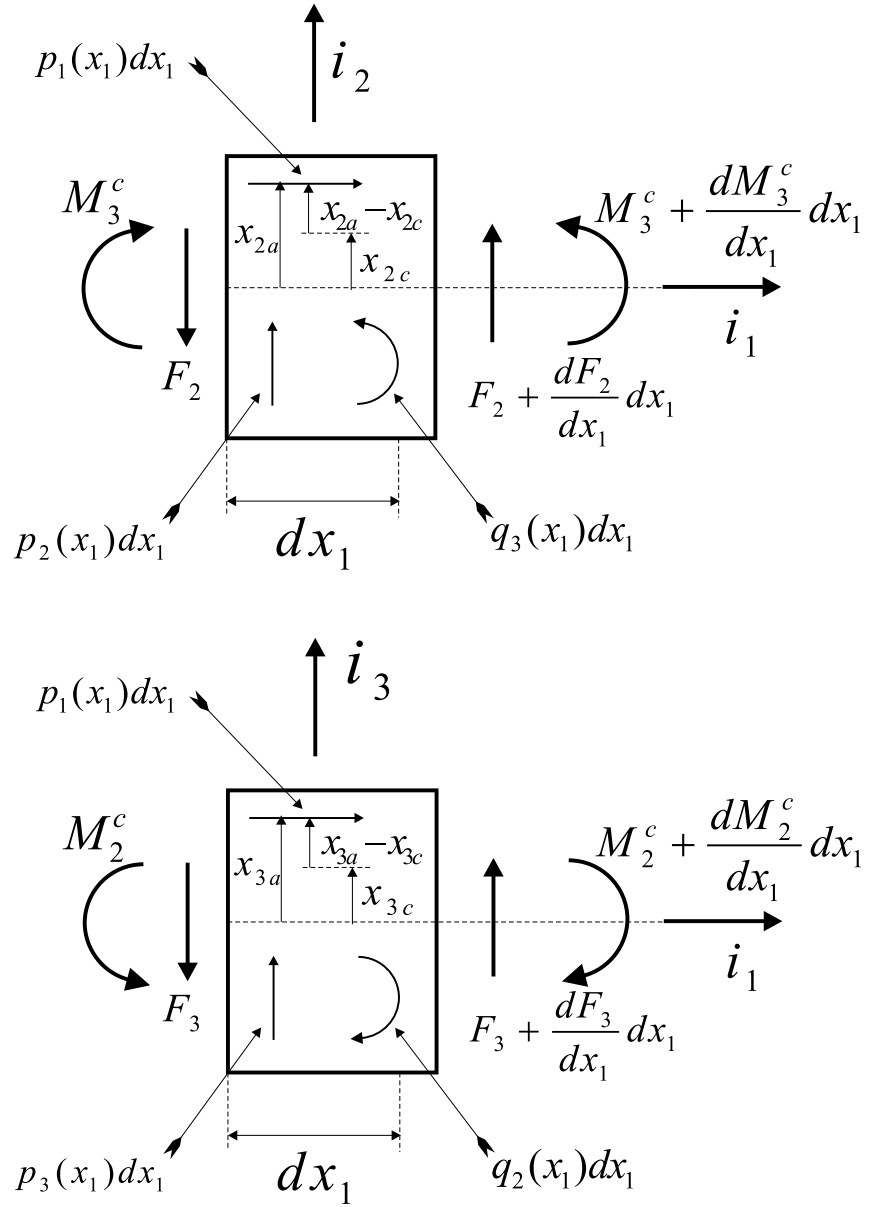


Figure 4.3: Free body diagram for the transverse shear forces and bending moments.

$$\frac{d^2 M_3^c}{dx_1^2} = p_2(x_1) + \frac{d}{dx_1}[(x_{2a} - x_{2c})p_1(x_1) - q_3(x_1)]. \quad (4.17)$$

4.4 Governing equations

The governing equations of the problem are obtained by introducing the sectional constitutive laws eq. (4.7) into the equilibrium equations, eqs. (4.11), (4.16), and (4.17), then using the definition of the sectional strains (3.7) to find

$$\frac{d}{dx_1} \left[S \frac{dU_1}{dx_1} - S_2^c \frac{d^2 U_2}{dx_1^2} - S_3^c \frac{d^2 U_3}{dx_1^2} \right] = -p_1 \quad (4.18)$$

$$\frac{d^2}{dx_1^2} \left[-S_2^c \frac{dU_1}{dx_1} + I_{33}^c \frac{d^2 U_2}{dx_1^2} + I_{23}^c \frac{d^2 U_3}{dx_1^2} \right] = p_2 + \frac{d}{dx_1}[(x_{2a} - x_{2c})p_1 - q_3] \quad (4.19)$$

$$\frac{d^2}{dx_1^2} \left[-S_3^c \frac{dU_1}{dx_1} + I_{23}^c \frac{d^2 U_2}{dx_1^2} + I_{22}^c \frac{d^2 U_3}{dx_1^2} \right] = p_3 + \frac{d}{dx_1}[(x_{3a} - x_{3c})p_1 + q_2] \quad (4.20)$$

The boundary conditions at the root of the beam are:

$$U_1 = U_2 = U_3 = 0; \quad \frac{dU_2}{dx_1} = \frac{dU_3}{dx_1} = 0; \quad (4.21)$$

which corresponds to vanishing displacements and slopes at the root of the beam. At the tip of the beam, the boundary conditions deal with the applied tip loads and moments:

$$\begin{aligned} F_1 &= P_1; & F_2 &= P_2; & F_3 &= P_3; \\ M_3^c &= Q_3 - (x_{2a} - x_{2c}) P_1; & M_2^c &= Q_2 + (x_{3a} - x_{3c}) P_1. \end{aligned} \quad (4.22)$$

Introducing the sectional constitutive laws (4.7) and using the definition of the sectional strains eq. (3.7) yields the boundary conditions expressed in terms of displacements as:

$$\begin{aligned} S \frac{dU_1}{dx_1} - S_2^c \frac{d^2 U_2}{dx_1^2} - S_3^c \frac{d^2 U_3}{dx_1^2} &= P_1; \\ -S_2^c \frac{dU_1}{dx_1} + I_{33}^c \frac{d^2 U_2}{dx_1^2} + I_{23}^c \frac{d^2 U_3}{dx_1^2} &= Q_3 - (x_{2a} - x_{2c}) P_1; \\ -S_3^c \frac{dU_1}{dx_1} + I_{23}^c \frac{d^2 U_2}{dx_1^2} + I_{22}^c \frac{d^2 U_3}{dx_1^2} &= -Q_2 - (x_{3a} - x_{3c}) P_1; \\ -\frac{d}{dx_1} \left[-S_2^c \frac{dU_1}{dx_1} + I_{33}^c \frac{d^2 U_2}{dx_1^2} + I_{23}^c \frac{d^2 U_3}{dx_1^2} \right] &= P_2 - [(x_{2a} - x_{2c})p_1 - q_3]_L; \\ -\frac{d}{dx_1} \left[-S_3^c \frac{dU_1}{dx_1} + I_{23}^c \frac{d^2 U_2}{dx_1^2} + I_{22}^c \frac{d^2 U_3}{dx_1^2} \right] &= P_3 - [(x_{3a} - x_{3c})p_1 + q_2]_L. \end{aligned} \quad (4.23)$$

The governing equations of the problem are in the form of the three coupled differential equations (4.18), (4.19), and (4.20), for the three sectional displacements U_1 , U_2 , and U_3 . The equations are second order in the axial displacement U_1 , and fourth order in the transverse displacements U_2 , and U_3 . There are ten associated boundary conditions, five at each end of the beam (4.21) and (4.23). Boundary conditions corresponding to various end configurations can be easily derived, as described in section (3.5.4).

4.5 Decoupling the three-dimensional problem

The governing equations described in the previous section are fully coupled equations, and as such present two major drawbacks: first, they are fairly difficult to solve, and second, their solution is difficult to interpret in physical terms. Indeed, as pointed out earlier, an axial displacement can result from an applied axial load but also from applied bending moments. Similarly, transverse displacements can result from applied bending moments but also from applied axial loads.

It is important to note that the theory developed in the previous section is very general. In particular, the coordinates (x_{2c}, x_{3c}) of the centroid are as yet arbitrary, as are the orientations of the axes $\mathbf{i}_2$ and $\mathbf{i}_3$ which can be arbitrarily rotated within the plane of the cross-section, i.e. about $\mathbf{i}_1$. The question to be raised in this section is whether a judicious choice of the location of the centroid and of the axes orientation can simplify the governing equations of the problem.

4.5.1 Definition of the centroid

Consider first the sectional stiffnesses S_2^c and S_3^c defined in (4.8). They can be made to vanish by an appropriate choice of x_{2c} and x_{3c} , indeed:

$$S_2^c = \int_{\Omega} E(x_2 - x_{2c}) d\Omega = 0; \quad S_3^c = \int_{\Omega} E(x_3 - x_{3c}) d\Omega = 0; \quad (4.24)$$

imply

$$x_{2c} = \frac{1}{S} \int_{\Omega} E x_2 d\Omega; \quad x_{3c} = \frac{1}{S} \int_{\Omega} E x_3 d\Omega, \quad (4.25)$$

which define the coordinates of the centroid. The constitutive laws for the cross-section (4.7) can be inverted to find

$$\begin{bmatrix} \epsilon_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix} = \begin{bmatrix} 1/S & 0 & 0 \\ 0 & I_{33}^c/\Delta & I_{23}^c/\Delta \\ 0 & I_{23}^c/\Delta & I_{22}^c/\Delta \end{bmatrix} \begin{bmatrix} F_1 \\ M_2^c \\ M_3^c \end{bmatrix}, \quad (4.26)$$

where $\Delta = I_{22}^c I_{33}^c - I_{23}^c I_{23}^c$. The axial stress (4.6) is now expressed in terms of the stress resultants

$$\sigma_1 = E \left[\frac{F_1}{S} + (x_3 - x_{3c}) \frac{I_{33}^c M_2^c + I_{23}^c M_3^c}{\Delta} - (x_2 - x_{2c}) \frac{I_{23}^c M_2^c + I_{22}^c M_3^c}{\Delta} \right] \quad (4.27)$$

4.5.2 Definition of the principal axes of bending

Consider now the sectional stiffness I_{23}^c defined in (4.10). It can be made to vanish by an appropriate choice of the axes orientation. By definition, the *principal axes of bending* are such that

$$I_{23}^c = \int_{\Omega} E (x_2 - x_{2c})(x_3 - x_{3c}) d\Omega = 0. \quad (4.28)$$

The procedure for determining the orientation of the principal axes of bending is detailed in section (4.6). Let $\mathbf{i}_2^*$ and $\mathbf{i}_3^*$ be the orientations of the principal axes of bending and $\mathcal{S}^*$ a

Cartesian reference frame defined by $\mathbf{i}_1^*$, $\mathbf{i}_2^*$, and $\mathbf{i}_3^*$. Of course, the axis of the beam remains unchanged, and $\mathbf{i}_1^* = \mathbf{i}_1$. The notation $(\cdot)^*$ will be used to indicate quantities measured in $\mathcal{S}^*$.

In this frame of reference, the constitutive laws for the cross-section (4.7) take the following, fully decoupled form

$$\epsilon_1 = \frac{F_1^*}{S^*}; \quad \kappa_2^* = \frac{M_2^{*c}}{I_{22}^{*c}}; \quad \kappa_3^* = \frac{M_3^{*c}}{I_{33}^{*c}}. \quad (4.29)$$

The corresponding axial stress distribution (4.6) becomes

$$\sigma_1^* = E \left[\frac{F_1^*}{S^*} + (x_3^* - x_{3c}^*) \frac{M_2^{*c}}{I_{22}^{*c}} - (x_2^* - x_{2c}^*) \frac{M_3^{*c}}{I_{33}^{*c}} \right] \quad (4.30)$$

4.5.3 Decoupled governing equations

The appropriate definition of the location of the centroid (4.24), and orientation of the orientation of the principal axes of bending (4.28) considerably simplifies the governing equation of the problem (4.18) to (4.20) which now decouple into three independent equations, describing the axial and bending behaviors of the beam.

The axial problem

First, the extensional problem is governed by the following equation

$$\frac{d}{dx_1^*} \left[S^* \frac{dU_1^*}{dx_1^*} \right] = -p_1^*; \quad (4.31)$$

subjected to the following boundary conditions at the root and tip of the beam

$$@x_1^* = 0 : \quad U_1^* = 0; \quad @x_1^* = L : \quad S^* \frac{dU_1^*}{dx_1^*} = P_1^*.$$

This extensional problem is identical to that discussed in section (3.4). Note that $S = S^*$ since the axial stiffness is not affected by a rotation of axes $\mathbf{i}_2$ and $\mathbf{i}_2$.

The first bending problem

Second, the bending problem in the pane ($\mathbf{i}_1^*$, $\mathbf{i}_2^*$) is governed by the following differential equation

$$\frac{d^2}{dx_1^{*2}} \left[I_{33}^{*c} \frac{d^2 U_2^*}{dx_1^{*2}} \right] = p_2^* + \frac{d}{dx_1^*} [(x_{2a}^* - x_{2c}^*) p_1^* - q_3^*] \quad (4.32)$$

subjected to the following boundary conditions at the root:

$$@x_1^* = 0 : \quad U_2^* = 0; \quad \frac{dU_2^*}{dx_1^*} = 0;$$

and at the tip:

$$\begin{aligned} @x_1^* = L : \quad I_{33}^{*c} \frac{d^2 U_2^*}{dx_1^{*2}} &= Q_3^* - (x_{2a}^* - x_{2c}^*) P_1^*; \\ -\frac{d}{dx_1^*} \left[I_{33}^{*c} \frac{d^2 U_2^*}{dx_1^{*2}} \right] &= P_2^* - [(x_{2a}^* - x_{2c}^*) p_1^* - q_3^*]_L. \end{aligned}$$

The second bending problem

Finally, the bending problem in the plane ($\mathbf{i}_1^*$, $\mathbf{i}_3^*$) with the following governing equation

$$\frac{d^2}{dx_1^{*2}} \left[I_{22}^{*c} \frac{d^2 U_3^*}{dx_1^{*2}} \right] = p_3^* + \frac{d}{dx_1^*} [(x_{3a}^* - x_{3c}^*) p_1^* + q_2^*] \quad (4.33)$$

subjected to the following boundary conditions at the root:

$$@x_1^* = 0 : \quad U_3^* = 0; \quad \frac{dU_3^*}{dx_1^*} = 0;$$

and at the tip:

$$\begin{aligned} @x_1^* = L : \quad I_{22}^{*c} \frac{d^2 U_3^*}{dx_1^{*2}} &= -Q_2^* - (x_{3a}^* - x_{3c}^*) P_1^*; \\ -\frac{d}{dx_1^*} \left[I_{22}^{*c} \frac{d^2 U_3^*}{dx_1^{*2}} \right] &= P_3^* - [(x_{3a}^* - x_{3c}^*) p_1^* + q_2^*]_L. \end{aligned}$$

Note that the two bending problem are identical problems written in the two orthogonal planes defined by the principal axes of bending. Each bending problem is identical to the bending problems discussed in section (3.5).

If the principal axes of bending are selected with their origin at the centroid, then $x_{2c}^* = x_{3c}^* = 0$ in the above equations. It is clear that the rotation of the axes to the principal directions takes place about axis $\mathbf{i}_1$. Hence, $\mathbf{i}_1^* = \mathbf{i}_1$, $x_1^* = x_1$, $U_1^* = U_1$, etc. The notational difference is made to emphasize the fact that all quantities in the decoupled equations are measured along the principal axes of bending.

4.6 The principal axes of bending

Consider an arbitrary set of axes $\mathbf{i}_2$, $\mathbf{i}_3$ with their origin at the centroid of the section, as depicted in fig. 4.4. Consider now a new set of axes $\mathbf{i}_2^*$, $\mathbf{i}_3^*$ defined by rotating the first set of axes by an angle α^* . Denoting (x_2, x_3) and (x_2^*, x_3^*) the coordinates of a point P measured along $\mathbf{i}_2$, $\mathbf{i}_3$, and $\mathbf{i}_2^*$, $\mathbf{i}_3^*$, respectively, the following transformations are readily obtained

$$x_2^* = x_2 \cos \alpha^* + x_3 \sin \alpha^*; \quad x_3^* = -x_2 \sin \alpha^* + x_3 \cos \alpha^*. \quad (4.34)$$

The centroidal bending stiffnesses in the $\mathbf{i}_2^*$, $\mathbf{i}_3^*$ system can be computed using eq. (4.9) to find:

$$I_{22}^{*c} = \int_{\Omega} E (-x_2 \sin \alpha^* + x_3 \cos \alpha^*)^2 d\Omega;$$

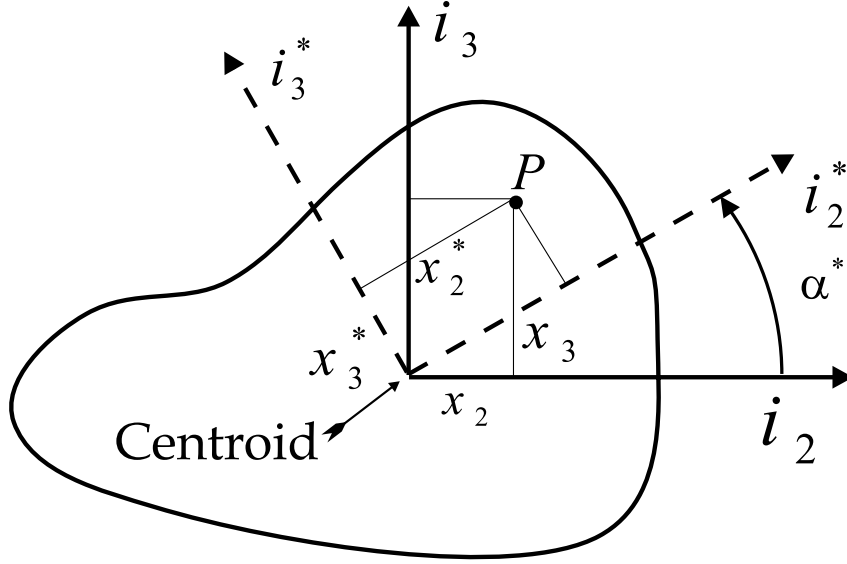


Figure 4.4: Rotation of the axes of the cross-section.

$$I_{33}^{*c} = \int_{\Omega} E (x_2 \cos \alpha^* + x_3 \sin \alpha^*)^2 d\Omega;$$

$$I_{23}^{*c} = \int_{\Omega} E (x_2 \cos \alpha^* + x_3 \sin \alpha^*)(-x_2 \sin \alpha^* + x_3 \cos \alpha^*) d\Omega.$$

Note that the coordinates of the centroid are zero since the axes are centered at the centroid. Expanding these expressions gives:

$$I_{22}^{*c} = I_{22}^c \cos^2 \alpha^* + I_{33}^c \sin^2 \alpha^* - 2I_{23}^c \sin \alpha^* \cos \alpha^*;$$

$$I_{33}^{*c} = I_{22}^c \sin^2 \alpha^* + I_{33}^c \cos^2 \alpha^* + 2I_{23}^c \sin \alpha^* \cos \alpha^*;$$

$$I_{23}^{*c} = (I_{22}^c - I_{33}^c) \sin \alpha^* \cos \alpha^* + I_{23}^c (\cos^2 \alpha^* - \sin^2 \alpha^*).$$

With the help of basic trigonometric identities, these expressions are rewritten as

$$I_{22}^{*c} = \frac{I_{33}^c + I_{22}^c}{2} - \frac{I_{33}^c - I_{22}^c}{2} \cos 2\alpha^* - I_{23}^c \sin 2\alpha^*; \quad (4.35)$$

$$I_{33}^{*c} = \frac{I_{33}^c + I_{22}^c}{2} + \frac{I_{33}^c - I_{22}^c}{2} \cos 2\alpha^* + I_{23}^c \sin 2\alpha^*; \quad (4.36)$$

$$I_{23}^{*c} = -\frac{I_{33}^c - I_{22}^c}{2} \sin 2\alpha^* + I_{23}^c \cos 2\alpha^*. \quad (4.37)$$

By definition (4.28), the principal axes of bending $\mathbf{i}_2^*$, $\mathbf{i}_3^*$ are such that $I_{23}^{*c} = 0$. Eq. (4.37) yields the following equation for α^* :

$$\tan 2\alpha^* = \frac{2I_{23}^c}{I_{33}^c - I_{22}^c}. \quad (4.38)$$

This equation gives the orientation of the principal axes of bending $\mathbf{i}_2^*$, $\mathbf{i}_3^*$ with respect to $\mathbf{i}_2$, $\mathbf{i}_3$. However, this definition is ambiguous as two solutions are possible for α^* that differ of 90° . To determine the principal axes of bending unequivocally it is convenient to use the following relationships

$$\sin 2\alpha^* = \frac{I_{23}^c}{\Delta}; \quad \cos 2\alpha^* = \frac{I_{33}^c - I_{22}^c}{2\Delta}, \quad (4.39)$$

where

$$\Delta = \sqrt{\left(\frac{I_{33}^c - I_{22}^c}{2}\right)^2 + (I_{23}^c)^2} \quad (4.40)$$

This is clearly equivalent to (4.38), but give a unique solution for α^* . The principal centroidal bending stiffnesses are now computed by introducing the orientation of the principal axes (4.39) into eqs. (4.35) and (4.36), to find

$$I_{22}^{*c} = \frac{I_{33}^c + I_{22}^c}{2} - \Delta; \quad I_{33}^{*c} = \frac{I_{33}^c + I_{22}^c}{2} + \Delta. \quad (4.41)$$

The angle α^* computed from (4.39) gives the orientation of the principal axis of bending $\mathbf{i}_2^*$ about which the principal centroidal bending stiffness I_{22}^{*c} is minimum.

In summary, the orientation of the principal axes of bending is obtained according to the following procedure.

1. Compute the centroid of the section using (4.25);
2. Compute the centroidal bending stiffnesses in an arbitrary coordinate system using eqs. (4.9) and (4.10);
3. Compute the orientation of the principal axes of bending using eq. (4.39);
4. Compute the principal centroidal bending stiffnesses using eq. (4.41).

It is interesting to note that the principal axes of bending are axes about which the bending stiffnesses are extremal: minimum about $\mathbf{i}_2$, and maximum about $\mathbf{i}_2$. Indeed, the bending stiffness is expressed in terms of α^* in (4.35). The minimum value of $I_{22}^{*c}(\alpha^*)$ occurs when its derivative with respect to α^* vanishes:

$$\frac{\partial I_{22}^{*c}}{\partial \alpha^*} = \frac{I_{33}^c - I_{22}^c}{2} 2 \sin 2\alpha^* - I_{23}^c 2 \cos 2\alpha^* = 0. \quad (4.42)$$

This condition is identical to (4.38). Similarly, the value of α^* which maximizes $I_{33}^{*c}(\alpha^*)$ is the very same value once more. In summary, the principal axes of bending are such that I_{23}^{*c} vanishes, and the corresponding bending stiffnesses are extremal.

4.7 Summary of the three-dimensional beam theory

Three-dimensional beam problems can be solved by following either of the two equivalent approaches described here.

□ Approach 1

1. Select an arbitrary set of axes $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, and project all applied load along these axes.
2. Compute the sectional stiffness coefficients (4.8), (4.9), and (4.10).
3. Solve the three coupled governing differential equations (4.18) to (4.20), subjected to the boundary conditions (4.21) and (4.23).

□ Approach 2

1. Compute the location of the centroid using (4.25).
2. Compute the orientation of the principal axes of bending $\mathbf{i}_1^*, \mathbf{i}_2^*, \mathbf{i}_3^*$, and the principal centroidal bending stiffnesses according to the procedure described in section (4.6),
3. Project all applied load along the principal axes of bending.
4. Solve three independent problems: an extensional problem (4.31) and two bending problems (4.32) and (4.33), subjected to the appropriate boundary conditions.

The two approaches will give identical results. The unknowns of the problem in the first approach are the displacements U_1, U_2 , and U_3 along an arbitrary set of axes, whereas the displacements U_1^*, U_2^* , and U_3^* along the principal axes of bending are the unknown of the second approach. The solution of the three *coupled* differential equations of the first approach is, in general, quite difficult to obtain. In the second approach, additional work, namely the computations of the centroid location and principal axes of bending orientation, is initially required. However, the solution phase then reduces to solving three *decoupled* differential equations.

4.8 Example

To demonstrate the use of the two approaches to three-dimensional beam problems a simple problem will be solved using both approaches, showing that identical results are obtained. Consider the uniform cantilevered beam subjected to a uniform transverse load p_0 as depicted in fig. 4.5. The beam is thin walled, i.e. $t/a \ll 1$, and made of a homogeneous material of Young's modulus E .

4.8.1 Approach 1

In this approach the first step is to select an arbitrary axis system: an axis system $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ that conveniently represents the geometry of the problem is shown in fig. 4.5. The next step is to compute the various sectional stiffnesses. The axial stiffness is computed first using (4.8)

$$S = E[at + 2at + at] = 4atE.$$

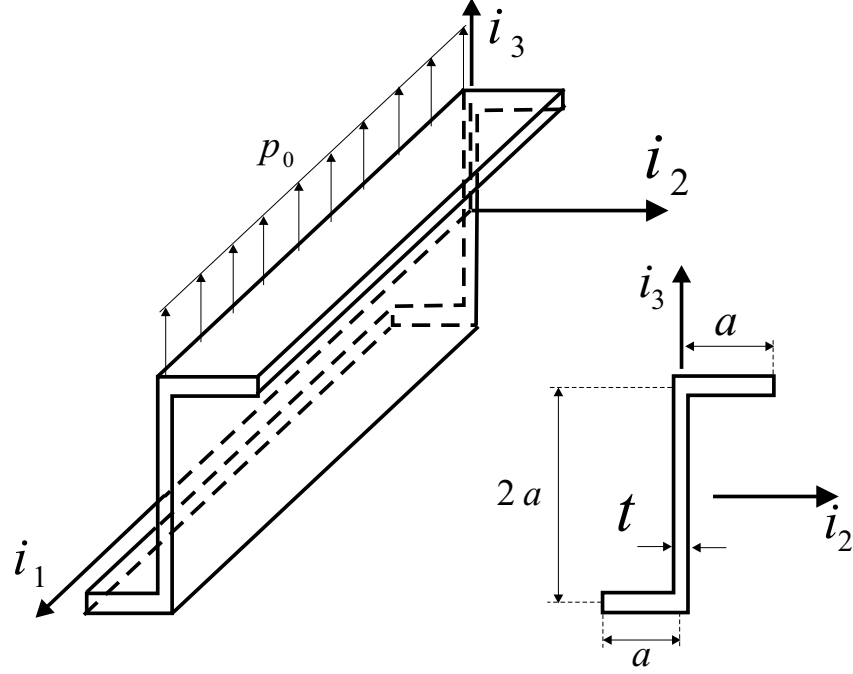


Figure 4.5: Cantilevered beam under a uniform transverse load.

In this approach the centroid location is arbitrary, and conveniently chosen as $x_{2c} = x_{3c} = 0$. Hence,

$$S_2^c = E \left[at \left(-\frac{a}{2} \right) + 2at(0) + at \left(\frac{a}{2} \right) \right] = 0;$$

$$S_3^c = E \left[at(-a) + 2at(0) + at(a) \right] = 0.$$

The bending stiffnesses are compute from (4.9)

$$I_{22}^c = E \left[\left(\frac{at^3}{12} + at a^2 \right) + \frac{t(2a)^3}{12} + \left(\frac{at^3}{12} + at a^2 \right) \right] \approx \frac{8a^3tE}{3};$$

$$I_{33}^c = E \left[\left(\frac{ta^3}{12} + at \frac{a^2}{4} \right) + \frac{2a(t)^3}{12} + \left(\frac{ta^3}{12} + at \frac{a^2}{4} \right) \right] \approx \frac{2a^3tE}{3},$$

where the thin wall approximation $t/a \ll 1$ was used. Finally the cross bending stiffness is obtained form (4.10)

$$I_{23}^c = E \left[at \left(-\frac{a}{2} \right) (-a) + 2at(0)(0) + at \left(\frac{a}{2} \right) (a) \right] = a^3tE.$$

Though the selected axis system conveniently describes the geometry of the problem, it does not coincide with the principal axes of bending which are characterized by a vanishing cross bending stiffness.

The third step of this approach is the solution of the governing equations which write

$$S \frac{d^2 U_1}{dx_1^2} = 0;$$

$$I_{33}^c \frac{d^4 U_2}{dx_1^4} + I_{23}^c \frac{d^4 U_3}{dx_1^4} = 0; \quad I_{23}^c \frac{d^4 U_2}{dx_1^4} + I_{22}^c \frac{d^4 U_3}{dx_1^4} = p_0.$$

The boundary conditions at the root are given as (4.21), and at the tip

$$S \frac{dU_1}{dx_1} = 0;$$

$$\begin{aligned} I_{33}^c \frac{d^2 U_2}{dx_1^2} + I_{23}^c \frac{d^2 U_3}{dx_1^2} &= 0; & I_{23}^c \frac{d^2 U_2}{dx_1^2} + I_{22}^c \frac{d^2 U_3}{dx_1^2} &= 0; \\ I_{33}^c \frac{d^3 U_2}{dx_1^3} + I_{23}^c \frac{d^3 U_3}{dx_1^3} &= 0; & I_{23}^c \frac{d^3 U_2}{dx_1^3} + I_{22}^c \frac{d^3 U_3}{dx_1^3} &= 0. \end{aligned}$$

The first equation is decoupled from the last two. Its solution is $U_1 = 0$. The last two equations are coupled but a simple algebraic manipulation yields:

$$\frac{d^4 U_2}{dx_1^4} = -\frac{I_{23}^c p_0}{I_{22}^c I_{33}^c - I_{23}^c I_{23}^c} = -\frac{9p_0}{7a^3 t E}; \quad \frac{d^4 U_3}{dx_1^4} = \frac{I_{33}^c p_0}{I_{22}^c I_{33}^c - I_{23}^c I_{23}^c} = \frac{6p_0}{7a^3 t E}.$$

Proceeding in a similar manner for the boundary conditions yields

$$@x_1 = 0 : U_2 = \frac{dU_2}{dx_1} = 0; \quad U_3 = \frac{dU_3}{dx_1} = 0;$$

$$@x_1 = l : \frac{dU_2^2}{dx_1^2} = \frac{dU_2^3}{dx_1^3} = 0; \quad \frac{dU_3^2}{dx_1^2} = \frac{dU_3^3}{dx_1^3} = 0.$$

Solving these two decoupled equations gives the solution of the problem

$$U_2(x_1) = -\frac{9p_0}{7a^3 t E} \frac{L^4}{24} \left[\left(\frac{x_1}{L}\right)^4 - 4\left(\frac{x_1}{L}\right)^3 + 6\left(\frac{x_1}{L}\right)^2 \right]; \quad (4.43)$$

$$U_3(x_1) = \frac{6p_0}{7a^3 t E} \frac{L^4}{24} \left[\left(\frac{x_1}{L}\right)^4 - 4\left(\frac{x_1}{L}\right)^3 + 6\left(\frac{x_1}{L}\right)^2 \right]; \quad (4.44)$$

The displacements at the tip of the beam are

$$U_{2tip} = -\frac{9}{56} \frac{p_0 L^4}{a^3 t E}; \quad U_{3tip} = \frac{6}{56} \frac{p_0 L^4}{a^3 t E}.$$

4.8.2 Approach 2

In this approach, the first step is to compute the location of the centroid using (4.25), to find

$$x_{2c} = \frac{E \left[at(-\frac{a}{2}) + 2at(0) + at(\frac{a}{2}) \right]}{4atE} = 0; \quad x_{3c} = \frac{E \left[at(-a) + 2at(0) + at(a) \right]}{4at} = 0.$$

The centroid happens to be located at the origin of the axes in this example. The next step is the computation of the orientation of the principal axes of bending. Equation (4.39) yields

$$\sin 2\alpha^* = \frac{a^3 t E}{a^3 t E \sqrt{2}} = \frac{1}{\sqrt{2}}; \quad \cos 2\alpha^* = \frac{-a^3 t E}{a^3 t E \sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

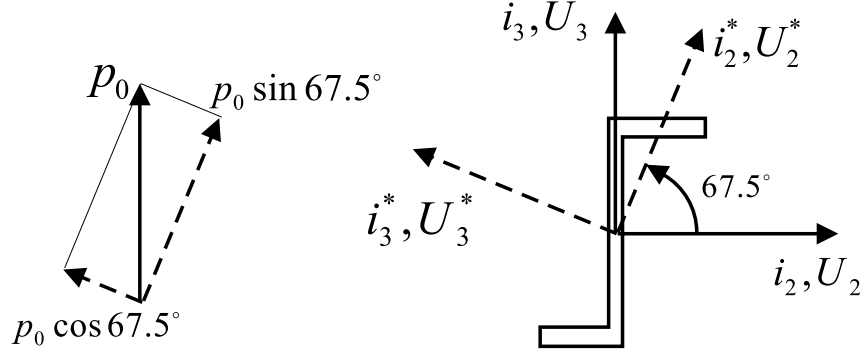


Figure 4.6: The principal axes of bending for the thin walled section.

The principal axis of bending $\mathbf{i}_2^*$ is oriented at a $\alpha^* = 67.5^\circ$ angle with respect to axis $\mathbf{i}_2$ as shown in fig. 4.6. The principal centroidal bending stiffnesses are found from eq. (4.41)

$$I_{22}^{*c} = \frac{5a^3tE}{3} - a^3tE\sqrt{2} = \left(\frac{5}{3} - \sqrt{2}\right)a^3tE; \quad I_{33}^{*c} = \left(\frac{5}{3} + \sqrt{2}\right)a^3tE.$$

The applied load are now projected along the directions of the principal axes of bending to find

$$p_2^* = p_0 \cos 22.5^\circ; \quad p_3^* = p_0 \sin 22.5^\circ.$$

The last step consists of the solution of three independent problems. As in the first approach the extensional problem yields $U_1^* = 0$. The two decoupled bending problems are

$$I_{33}^{*c} \frac{d^4 U_2^*}{dx_1^{*4}} = p_0 \cos 22.5^\circ; \quad I_{22}^{*c} \frac{d^4 U_3^*}{dx_1^{*4}} = p_0 \sin 22.5^\circ.$$

subjected to the following boundary conditions at the root:

$$@x_1^* = 0 : \quad U_2^* = \frac{dU_2^*}{dx_1^*} = 0; \quad U_3^* = \frac{dU_3^*}{dx_1^*} = 0;$$

and at the tip:

$$@x_1^* = L : \quad \frac{d^2 U_2^*}{dx_1^{*2}} = \frac{d^3 U_2^*}{dx_1^{*3}} = 0; \quad \frac{d^2 U_3^*}{dx_1^{*2}} = \frac{d^3 U_3^*}{dx_1^{*3}} = 0.$$

The solution of these two decoupled equations is:

$$U_2^*(x_1^*) = -\frac{p_0 \cos 22.5^\circ}{I_{33}^{*c}} \frac{L^4}{24} \left[\left(\frac{x_1^*}{L}\right)^4 - 4\left(\frac{x_1^*}{L}\right)^3 + 6\left(\frac{x_1^*}{L}\right)^2 \right]; \quad (4.45)$$

$$U_3^*(x_1^*) = \frac{p_0 \sin 22.5^\circ}{I_{22}^{*c}} \frac{L^4}{24} \left[\left(\frac{x_1^*}{L}\right)^4 - 4\left(\frac{x_1^*}{L}\right)^3 + 6\left(\frac{x_1^*}{L}\right)^2 \right]. \quad (4.46)$$

The corresponding deflections at the tip of the beam become

$$U_{2tip}^* = -\frac{3 \cos 22.5^\circ}{24\left(\frac{5}{3} + \sqrt{2}\right)} \frac{p_0 L^4}{a^3 t E}; \quad U_{3tip}^* = \frac{3 \sin 22.5^\circ}{24\left(\frac{5}{3} - \sqrt{2}\right)} \frac{p_0 L^4}{a^3 t E}.$$

4.8.3 Discussion of the results

The two approaches give the displacements in two different coordinate system. The displacements U_2 and U_3 given by eqs. (4.43) and (4.44), respectively, are the components of displacements along the axes $\mathbf{i}_2$, $\mathbf{i}_3$, respectively, whereas the displacements U_2^* and U_3^* given by eqs. (4.45) and (4.46), respectively, are the components of displacements along the principal axes of bending $\mathbf{i}_2^*$, $\mathbf{i}_3^*$, respectively. However, these two results describe identical deformations of the beam. Indeed, fig. 4.6 shows that the two sets of displacement are related through the following expressions

$$U_2^* = U_2 \cos 67.5^\circ + U_3 \sin 67.5^\circ; \quad U_3^* = -U_2 \sin 67.5^\circ + U_3 \cos 67.5^\circ.$$

The displacement fields found in the two approaches identically satisfy this expression.

It is important to note that though the applied load acts in the $\mathbf{i}_3$ direction only, the beam displaces in *both* $\mathbf{i}_3$ and $\mathbf{i}_2$ directions. Actually the tip displacement component in the $\mathbf{i}_2$ direction is *larger* than that in the $\mathbf{i}_3$ direction. This is due to the fact that the bending behaviors in the $\mathbf{i}_1$ $\mathbf{i}_2$ and $\mathbf{i}_1$ $\mathbf{i}_3$ planes are coupled, as expressed by the coupled governing equations (4.18), (4.19), and (4.20).

This behavior is more easily understood when considering the results of the second approach. Indeed, the bending behaviors of the beam along the principal axes of bending are *decoupled*. This means that the load p_2^* applied along the $\mathbf{i}_2^*$ direction produces a displacement *along the $\mathbf{i}_2^*$ direction only*. Similarly, the load p_3^* applied along the $\mathbf{i}_3^*$ direction produces a displacement *along the $\mathbf{i}_3^*$ direction only*. The displacement along the $\mathbf{i}_2^*$ direction is fairly small because the bending stiffness I_{33}^{*c} that characterizes bending about the $\mathbf{i}_3^*$ axis is maximum. On the other hand, the displacement in the $\mathbf{i}_3^*$ direction is large because the bending stiffness I_{22}^{*c} that characterizes bending about the $\mathbf{i}_2^*$ axis is minimum. The resulting displacement U_3^* , when resolved along the axes $\mathbf{i}_2$ and $\mathbf{i}_3$, has the expected upward component, together with a *leftward* component. This explains the negative sign of the U_2 in (4.43).

4.8.4 Problems

Problem 4.1

Consider the thin-walled, L shaped cross-section of a beam as shown in fig. 4.7. Let $b = 0.25$ m, $h = 0.1$ m, and $t = 2.5 \times 10^{-3}$ m.

1. Find the location of the centroid of the section.
2. Find the orientation of the principal centroidal axes of bending.

Problem 4.2

Fig. 4.8 depicts a cantilevered beam subjected to a tip axial load P applied at the lower left edge of the section. The length of the beam is $L = 10b$, the upper and lower flanges are of length b and $3b/2$, respectively, the web is of height $h = 2b$ and the wall thickness $t = b/10$.

1. Determine the location of centroid of the section.

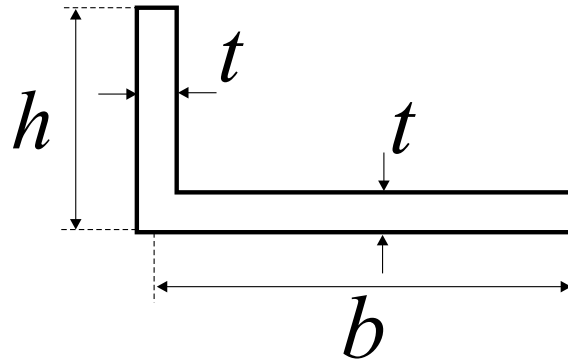
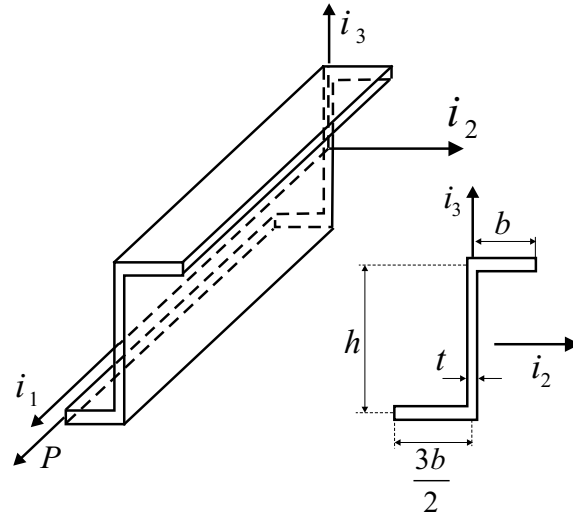
Figure 4.7: Thin-walled, L shaped cross-section.

Figure 4.8: Cantilevered beam with Z-section under tip axial load.

2. Determine the bending stiffnesses I_{22}^c , I_{33}^c , and I_{23}^c for the axis system shown on the figure.
3. Determine the orientation of the principal axes of bending, $\mathbf{i}_2^*$ and $\mathbf{i}_3^*$, and the principal centroidal bending stiffnesses I_{22}^{*c} , I_{33}^{*c} .
4. Solve this problem in the coordinate system shown on the figure to determine the displacements $U_1(x_1)$, $U_2(x_1)$, and $U_3(x_1)$, as well as the cross-sectional rotations $\Phi_2(x_1)$ and $\Phi_3(x_1)$.
5. Solve this problem in the coordinate system defined by the principal axes of bending to determine the displacements $U_1^*(x_1)$, $U_2^*(x_1)$, and $U_3^*(x_1)$, as well as the cross-sectional rotations $\Phi_2^*(x_1)$ and $\Phi_3^*(x_1)$.
6. Show that the two above solutions are identical.
7. Find the three components of displacement at the point of application of the load P , EbU_1/P , EbU_2/P , and EbU_3/P , respectively.

8. Find the axial stress distribution at the root of the beam. Plot this distribution over the root section. Where does the maximum axial stress occur? Use the nondimensional stress $b^2\sigma_1/P$.

Chapter 5

Torsion

In the previous chapters, the behavior of beams subjected to axial and transverse loads was studied in detail. In chapter 4, a fairly general, three dimensional loading was considered, with one important restriction: the beam is assumed to *bend without twisting*. However, twisting is often present in structures, and in fact, many important structural components are designed to primarily carry torsional loads.

Power transmission drive shafts are a prime example of structural components designed to carry a specific torque. Such components present solid, or thin-walled circular sections. Numerous other structural components are designed to carry a combination of axial, bending, and torsional loads. For instance, an aircraft wing must carry the bending and torsional moments generated by the aerodynamic forces. The behavior of structural components under torsional loads is the focus of this chapter.

5.1 Torsion of circular cylinders

Consider an infinitely long, homogeneous, solid or hollow circular cylinder subjected to end torques Q_1 of equal magnitude and opposite directions as depicted in fig. 5.1. The cross-section of the cylinder can be a circle of radius R , or a circular annulus of inner and outer radii R_i and R_o , respectively. This problem is characterized by two types of symmetries: first a circular symmetry about the $\mathbf{i}_1$ axis, and second, a symmetry with respect to any plane normal to $\mathbf{i}_1$.

These symmetries imply a number of constraints on the kinematics of the deformation of the system. First, all sections must remain circular and rotate about their own center. This is a direct consequence of the circular symmetry of the problem which implies that the deformation of all radial material lines must be identical. Second, the cross-section does not deform out-of-plane, in other words, the section does not warp out of its own plane. Indeed, such a deformation cannot take place in view of the symmetry about any cross-sectional plane. Third, any radial line of the cross-section must remain straight during deformation, since any curvature of this line would violate the symmetry of the problem about the sectional plane. Finally, the angle between two radial lines of the cross-section is unaffected by the deformation, as required by the circular symmetry of the problem.

In summary, each cross-section *rotates about its own center like a rigid disk*. This is the

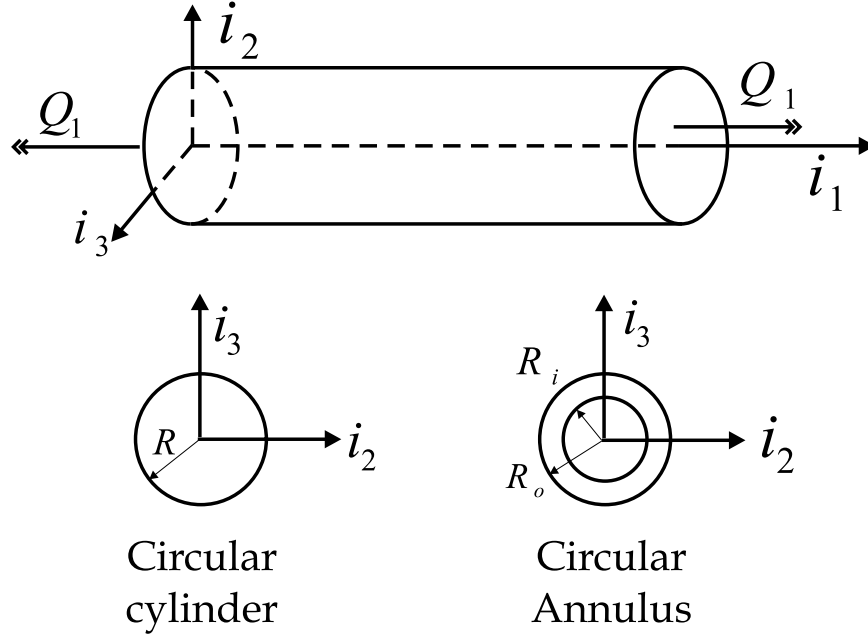


Figure 5.1: Circular cylinder with end torques.

only deformation compatible with the symmetries of the problem.

5.1.1 Kinematic description

Let $\Phi_1(x_1)$ be the rigid rotation of the cross-section. This rotation brings point A of the section to A' . Fig. 5.2 shows the polar coordinates r and α defining the position of point A . The sectional in-plane displacement field writes:

$$u_2(x_1, r, \alpha) = -r\Phi_1(x_1) \sin \alpha; \quad u_3(x_1, r, \alpha) = r\Phi_1(x_1) \cos \alpha. \quad (5.1)$$

The magnitude of the rigid body rotation is assumed to be small so that the segment AA' can be approximated by a straight line of length $r\Phi_1$. Since the cross-section does not deform out of its own plane, the axial displacement field must vanish, i.e. $u_1(x_1, x_2, x_3) = 0$. The transformation from polar to Cartesian coordinates is

$$x_2 = r \cos \alpha; \quad x_3 = r \sin \alpha. \quad (5.2)$$

The complete displacement field describing the torsion of circular cylinders expressed in Cartesian coordinates now becomes

$$u_1(x_1, x_2, x_3) = 0; \quad (5.3)$$

$$u_2(x_1, x_2, x_3) = -x_3\Phi_1(x_1); \quad u_3(x_1, x_2, x_3) = x_2\Phi_1(x_1). \quad (5.4)$$

The corresponding strain field is readily obtained as:

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1} = 0; \quad (5.5)$$

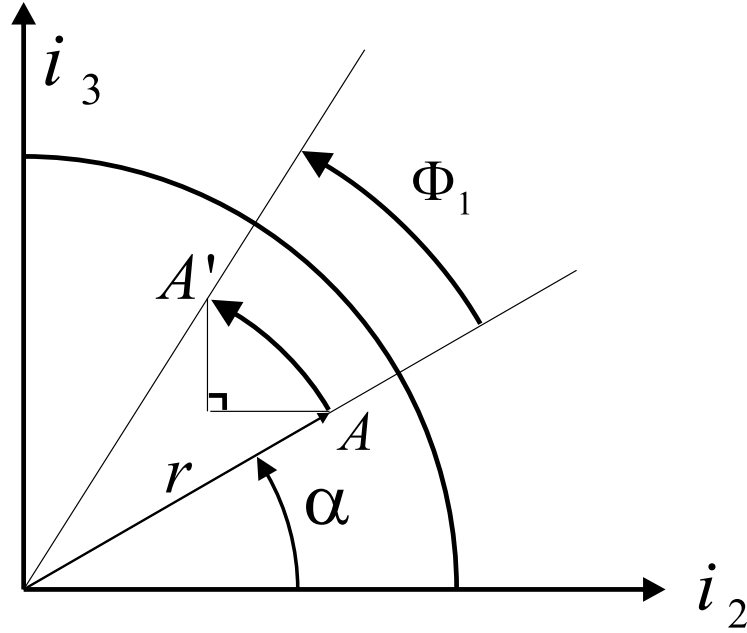


Figure 5.2: In-plane displacements for a circular cylinder.

$$\varepsilon_2 = \frac{\partial u_2}{\partial x_2} = 0; \quad \varepsilon_3 = \frac{\partial u_3}{\partial x_3} = 0; \quad \gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0; \quad (5.6)$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = -x_3 \kappa_1(x_1); \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = x_2 \kappa_1(x_1), \quad (5.7)$$

where the sectional twist rate is defined as

$$\kappa_1(x_1) = \frac{d\Phi_1}{dx_1}. \quad (5.8)$$

The axial strain field (5.5) vanishes because the section does not warp out-of-plane; and the in-plane-strain field (5.6) vanishes since the in-plane motion of the section is a rigid body rotation. During torsion, the only non vanishing strain components are the out-of-plane shearing strains, eq. (5.7). This strain field is not easily visualized in rectangular coordinates. The radial and circumferential shearing strain components in polar coordinates denoted γ_r and γ_α , respectively, are related to the Cartesian component as

$$\gamma_r = \gamma_{12} \cos \alpha + \gamma_{13} \sin \alpha; \quad \gamma_\alpha = -\gamma_{12} \sin \alpha + \gamma_{13} \cos \alpha. \quad (5.9)$$

Introducing (5.7) yields the out-of-plane shearing field in polar coordinates

$$\gamma_r(x_1, r, \alpha) = 0; \quad \gamma_\alpha(x_1, r, \alpha) = r \kappa_1(x_1). \quad (5.10)$$

It is now clear that the only non vanishing strain component is the circumferential shearing strain γ_α which is proportional to the twist rate κ_1 , and varies linearly from zero at the center of the section to its maximum value $R \kappa_1$ along the outer edge of the cylinder. It is of course independent of α , as required by the cylindrical symmetry of the problem.

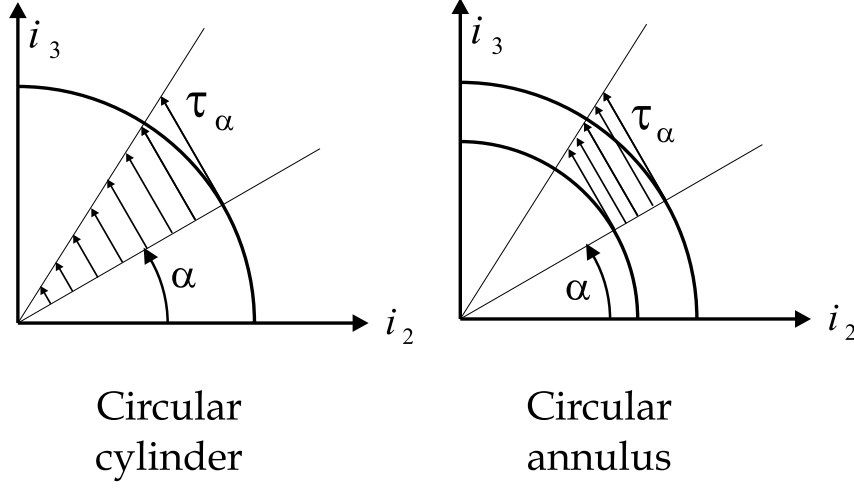


Figure 5.3: Distribution of circumferential shearing stress over the cross-section.

5.1.2 The stress field

Let the cylinder be made of a linear elastic material that obeys Hooke's law, eq. (1.91). In view of the strain field (5.7), the only non vanishing stress components are

$$\tau_{12} = -Gx_3 \kappa_1(x_1); \quad \tau_{13} = Gx_2 \kappa_1(x_1), \quad (5.11)$$

where G is the shearing modulus of the material. Once again, polar coordinates are more convenient to visualize the stress field which is readily obtained from eq. (5.10)

$$\tau_r(x_1, r, \alpha) = 0; \quad \tau_\alpha(x_1, r, \alpha) = Gr \kappa_1(x_1), \quad (5.12)$$

where τ_r and τ_α are the radial and circumferential shearing stress components, respectively. The distribution of the circumferential shearing stress over the cross-section is shown in fig. 5.3.

5.1.3 Sectional constitutive law

The torque acting on the cross-section at a given span-wise location is readily obtained by integrating the circumferential shearing stress τ_α multiplied by the moment arm r to find

$$M_1(x_1) = \int_{\Omega} \tau_\alpha r \, d\Omega. \quad (5.13)$$

Introducing the circumferential shearing stress, eq. (5.12) then yields

$$M_1(x_1) = \int_{\Omega} Gr^2 \kappa_1(x_1) \, d\Omega = J \kappa_1(x_1), \quad (5.14)$$

where the *torsional stiffness* of the section was defined as

$$J = \int_{\Omega} Gr^2 \, d\Omega. \quad (5.15)$$

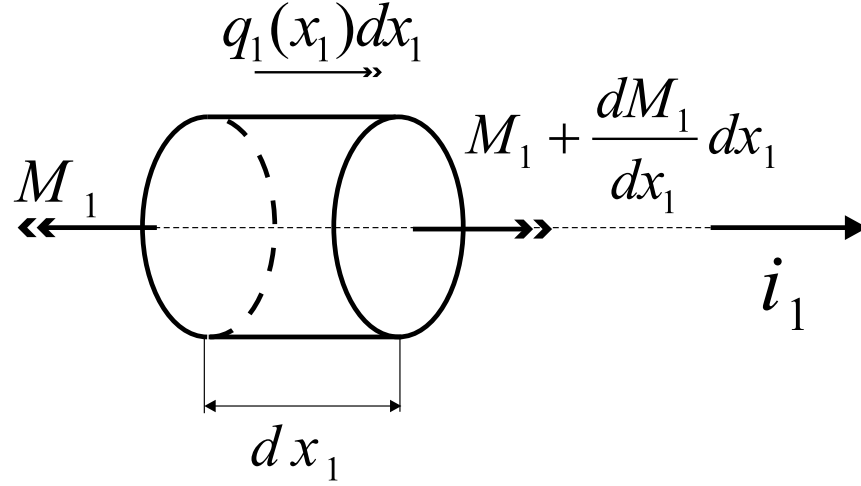


Figure 5.4: Torsional loads acting on an infinitesimal slice of the beam.

Relationship (5.14) is the constitutive law for the torsional behavior of the beam. It expresses the proportionality between the torque and the twist rate, with a constant of proportionality J called the torsional stiffness.

5.1.4 Equilibrium equations

The equations of equilibrium associated with the torsional behavior can be obtained by considering the infinitesimal slice of the cylinder of length dx_1 depicted in fig. 5.4. Summing all the moments about axis i_1 yields the torsional equilibrium equation

$$\frac{dM_1}{dx_1} = -q_1. \quad (5.16)$$

5.1.5 Governing equations

Finally, the governing equation for the torsional behavior of circular cylinders is obtained by introducing the torque (5.14) into the equilibrium equation (5.16) and recalling the definition of the twist rate

$$\frac{d}{dx_1} \left[J \frac{d\Phi_1}{dx_1} \right] = -q_1. \quad (5.17)$$

This second order differential equation can be solved for the twist distribution Φ_1 given the torque distribution q_1 .

Two boundary conditions are required for the solution of eq. (5.17), one at each end of the cylinder. Typical boundary conditions are:

1. A fixed (or clamped) end allows no rotation, i.e.:

$$\Phi_1 = 0; \quad (5.18)$$

2. A free (unloaded) end corresponds to $M_1 = 0$, which in view of eq. (5.14) write:

$$\frac{d\Phi_1}{dx_1} = 0; \quad (5.19)$$

3. Finally, if the end of the cylinder is subjected to a concentrated torque Q_1 , the boundary condition is $M_1 = Q_1$, which becomes:

$$J \frac{d\Phi_1}{dx_1} = Q_1. \quad (5.20)$$

5.1.6 The torsional stiffness

The torsional stiffness J of the section characterizes the stiffness of the cylinder when subjected to torsion. If the cylinder is made of a homogeneous material, the shearing modulus is identical at all points of the cross-section and can be factored out of integral (5.15) which is then easily evaluated in polar coordinates

$$J = G \int_0^{2\pi} \int_0^R r^2 r dr d\alpha = \frac{\pi}{2} G R^4. \quad (5.21)$$

For a circular tube the second integral extends from the inner radius R_i to the outer radius R_o to find

$$J = G \int_0^{2\pi} \int_{R_i}^{R_o} r^2 r dr d\alpha = \frac{\pi}{2} G (R_o^4 - R_i^4) \quad (5.22)$$

A common situation of great practical importance is that of a thin-walled circular tube. Let the mean radius of the tube be $R_m = (R_o + R_i)/2$, and the wall thickness $t = R_o - R_i$. The thin wall assumption implies

$$\frac{t}{R_m} \ll 1. \quad (5.23)$$

The torsional stiffness of the thin-walled tube then becomes

$$J = \frac{\pi}{2} G (R_o^2 + R_i^2) (R_o + R_i) (R_o - R_i) \approx 2\pi G R_m^3 t. \quad (5.24)$$

Consider now a thin-walled circular tube with several layers of different materials through the wall thickness, as depicted in fig. 5.5. Assuming the material to be homogeneous within each layer with a shearing modulus $G^{[i]}$ in layer i , the torsional stiffness becomes

$$J = \frac{\pi}{2} \sum_{i=1}^n G^{[i]} [(R^{[i+1]})^4 - (R^{[i]})^4].$$

For a thin-walled tube, each layer will be thin, and the above approximation can be used once again to find:

$$J = 2\pi \sum_{i=1}^n G^{[i]} t^{[i]} \left(\frac{R^{[i+1]} + R^{[i]}}{2} \right)^3. \quad (5.25)$$

From this expression it is clear that the torsional stiffness is a *weighted average* of the shearing moduli of the various layer. The weighting factor $t^{[i]} ((R^{[i+1]} + R^{[i]})/2)^3$ strongly biases the average in favor of the outermost layers.

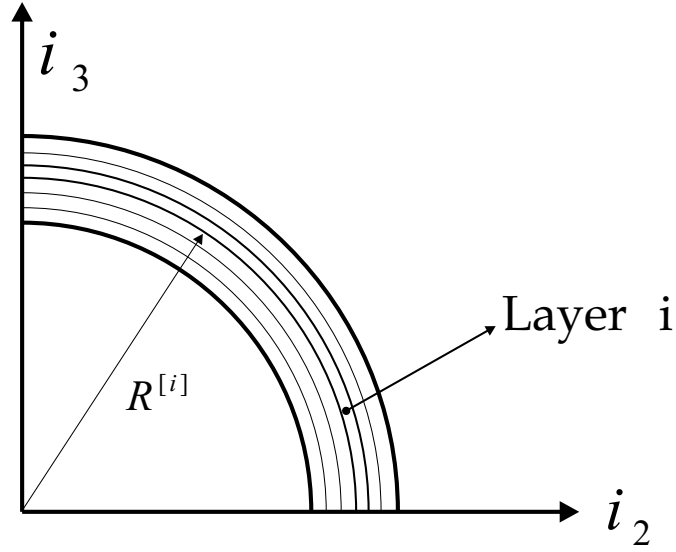


Figure 5.5: Thin-walled tube made of layered materials.

5.1.7 The shearing stress distribution

The local circumferential shearing stress can be related to the sectional torque by eliminating the twist rate between eqs. (5.12) and (5.14) to find

$$\tau_\alpha = G \frac{M_1(x_1)}{J} r. \quad (5.26)$$

This shearing stress distribution was depicted in fig. 5.3. The maximum shearing stress occurs at the outer edge of the solid cylinder and can be readily evaluated with the help of (5.21)

$$\tau_\alpha^{max} = \frac{2M_1(x_1)}{\pi R^3}. \quad (5.27)$$

For a circular tube, the maximum shearing stress occurs once more along the outer edge of the tube, and for (5.22), writes

$$\tau_\alpha^{max} = \frac{2R_o M_1(x_1)}{\pi(R_o^4 - R_i^4)}. \quad (5.28)$$

If the circular tube is thin-walled, the shearing stress is nearly uniform through the thickness of the wall and is

$$\tau_\alpha^{max} = \frac{M_1(x_1)}{2\pi R_m^2 t}. \quad (5.29)$$

In contrast, the shearing stress distribution for thin-walled sections with various layers will be nearly uniform within each layer

$$\tau_\alpha^{[i]} = G^{[i]} \frac{R^{[i+1]} + R^{[i]}}{2} \frac{M_1(x_1)}{J}, \quad (5.30)$$

where the torsional stiffness J was computed in eq. (5.25).

Once the local shear stress has been determined, a strength criterion is applied to determine whether the structure can sustain the applied loads. Combining the strength criterion, eq. (2.9), and eq. (5.27) yields $(GR/J) |M_1(x_1)| \leq \tau_{\text{allow}}$. Since the torque varies along the span of the beam, this condition must be checked at all point along the span. In practice, it is convenient to first determine the maximum torque denoted $M_{1 \text{ max}}$, then apply the strength criterion

$$\frac{GR}{J} |M_{1 \text{ max}}| \leq \tau_{\text{allow}}. \quad (5.31)$$

If the section consists of layers made of various materials, the strength of each layer will, in general, be different, and the strength criterion becomes

$$\frac{G^{[i]}r}{J} |M_{1 \text{ max}}| \leq \tau_{\text{allow}}^{[i]}, \quad (5.32)$$

where $\tau_{\text{allow}}^{[i]}$ is the allowable shear stresses for layer i . The strength criterion must be checked for each material layer.

5.1.8 The strain energy

Consider an infinitesimal slice of the beam acted upon by a torque M_1 . Under the effect of this torque, the end cross-sections of the slice rotate of a differential amount $d\Phi_1$. If this deformation is proportional to the applied moment, the work dW done by the torque as it increases from zero to its present value M_1 is

$$dW = \frac{1}{2} M_1 d\Phi_1 = \frac{1}{2} M_1 \frac{d\Phi_1}{dx_1} dx_1.$$

This work is given by the area under the torque-twist rate curve. Introducing the sectional constitutive law, eq. (5.14), and the curvature displacement relationship, eq. (5.8), leads to

$$dW = \frac{1}{2} J \kappa_1^2 dx_1. \quad (5.33)$$

The work done by the torque is stored in the slice of the beam in the form of strain energy, and the quantity

$$a(\kappa_1) = \frac{1}{2} J \kappa_1^2, \quad (5.34)$$

is known as the strain energy density function under twist rate. Eq. (5.33) expresses the fact that the work done by the torque equals the amount of strain energy stored in the slice of the beam. The total work done by the torque distribution in the beam is now

$$W = \frac{1}{2} \int_0^L J \kappa_1^2 dx_1 = A(\kappa_1), \quad (5.35)$$

and equal the total amount of strain energy stored in the beam, $A(\kappa_1)$.

Sometimes, it is preferable to express the deformation energy stored in the beam in terms of the torque by using eq. (5.14), to find

$$A(\kappa_1) = \int_0^L \frac{M_1^2}{2J} dx_1 = B(M_1). \quad (5.36)$$

$b(M_1) = M_1^2/2J$ is known as the stress energy density function. $B(M_1)$ is the total deformation energy stored in the beam expressed in terms of the torque, also called complementary energy.

5.1.9 Rational design of cylinders under torsion

The shearing stress distribution in a cylinder subjected to torsion was shown in fig. 5.3. Clearly, the material near the center of the cylinder is not used very efficiently since the shearing stress becomes very small in the central portion of the cylinder. A far more efficient design is the thin-walled tube. Indeed, the shearing stress becomes nearly uniform through the thickness of the wall, and all the material can be used at full capacity.

For a homogeneous thin walled tube, the mass of material per unit span is $m = 2\pi R_m t \rho$, where ρ is the material density. The torsional stiffness (5.24) now writes

$$J = \frac{m}{\rho} G R_m^2. \quad (5.37)$$

Consider two thin-walled tubes made of identical materials, with identical masses per unit span, and mean radii R_m and R'_m , respectively. The ratio of their torsional stiffnesses, noted J and J' , respectively write

$$\frac{J}{J'} = \left(\frac{R_m}{R'_m} \right)^2. \quad (5.38)$$

For identical masses of material, the torsional stiffness increases with the square of the mean radius. When subjected to identical torques, the shear stresses in the two tubes are noted τ_α and τ'_α , respectively. Their ratio is

$$\frac{\tau_\alpha}{\tau'_\alpha} = \frac{R'_m}{R_m}. \quad (5.39)$$

For identical masses of material, the shearing stress decays in proportion to the mean radius.

The ideal structure to carry torsional loads clearly is a thin-walled tube with a very large mean radius. There often exist practical limits on how large the mean radius can be. Furthermore, very thin-walled tubes can become unstable, a phenomenon called *torsional buckling*. This type of instability puts a limit on how thin the wall can be.

5.2 Strength under combining loading

Consider an aircraft propeller connected to a homogeneous, circular shaft. The engine applies a torque to the shaft resulting in the shear stress distribution described in section 5.1.7. On the other hand, the propeller creates a thrust that generates a uniform axial stress distribution over the cross-section. If the torque were to act alone, the strength criterion would write $\tau < \tau_{\text{allow}}$. If the axial force were to act alone, the corresponding criterion would be $\sigma < \sigma_{\text{allow}}$. The question is now: what is the proper strength criterion to be used when both axial and shear stresses are acting simultaneously?

5.2.1 Von Mises strength criterion

Consider an isotropic, homogeneous material subjected to a general three-dimensional state of stress. The following equivalent stress is defined

$$\sigma_{\text{eq}} = [\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - \sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11} - \sigma_{11}\sigma_{22} + 3(\tau_{23}^2 + \tau_{13}^2 + \tau_{12}^2)]^{1/2}. \quad (5.40)$$

Von Mises strength criterion postulates that under combined loading, the safe stress level is such that the equivalent stress is smaller than the allowable stress,

$$\sigma_{\text{eq}} \leq \sigma_{\text{allow}}. \quad (5.41)$$

At first, consider the case of a bar under an axial force. The sole non vanishing stress component is σ_{11} , the equivalent stress become $\sigma_{\text{eq}} = \sigma_{11}$, and the strength criterion writes $\sigma_{11} \leq \sigma_{\text{allow}}$. This result is identical to the strength criterion discussed in section 2.2.

Next consider the case of a material under pure plane shear, *i.e.* all stress components vanish except for τ_{12} . The equivalent stress become $\sigma_{\text{eq}} = \sqrt{3} \tau_{12}$, and the strength criterion $\tau_{12} \leq \sigma_{\text{allow}}/\sqrt{3}$. This important result implies that the allowable shear stress is related to the allowable axial stress as

$$\tau_{\text{allow}} = \frac{\sigma_{\text{allow}}}{\sqrt{3}}. \quad (5.42)$$

This result is found to be in excellent agreement with experimental measurements, providing an experimental verification of Von Mises strength criterion. The following examples describe various applications of this criterion.

Pressure vessel

In section 2.6, a pressure vessel subjected to internal pressure was shown develop both hoop stresses $\sigma_h = pR/e$ and axial stresses $\sigma_a = \sigma_h/2$. Since all other stress components vanish the equivalent stress become $\sigma_{\text{eq}} = [\sigma_h^2 + \sigma_a^2 - \sigma_h\sigma_a]^{1/2}$. Von Mises criterion now implies

$$\frac{\sqrt{3}}{2} \frac{pR}{e} \leq \sigma_{\text{allow}}, \text{ or } p \leq \frac{2}{\sqrt{3}} \frac{e\sigma_{\text{allow}}}{R} \approx 1.15 \frac{e\sigma_{\text{allow}}}{R}. \quad (5.43)$$

It is interesting to note that if the strength criterion was erroneously applied by taking into account the sole hoop stress (the maximum stress component) the safe service internal pressure would be $p \leq e\sigma_{\text{allow}}/R$, a more stringent condition.

Propeller shaft

Consider an aircraft propeller connected to a homogeneous, circular shaft of radius R . The engine applies a torque M_1 to the shaft and the propeller exerts a thrust F_1 . The corresponding stresses are

$$\tau = \frac{2M_1}{\pi R^3}, \text{ and } \sigma = \frac{F_1}{\pi R^2}, \quad (5.44)$$

respectively. Von Mises criterion then requires

$$\left[\left(\frac{F_1}{\pi R^2} \right)^2 + 3 \left(\frac{2M_1}{\pi R^3} \right)^2 \right]^{1/2} \leq \sigma_{\text{allow}} \quad (5.45)$$

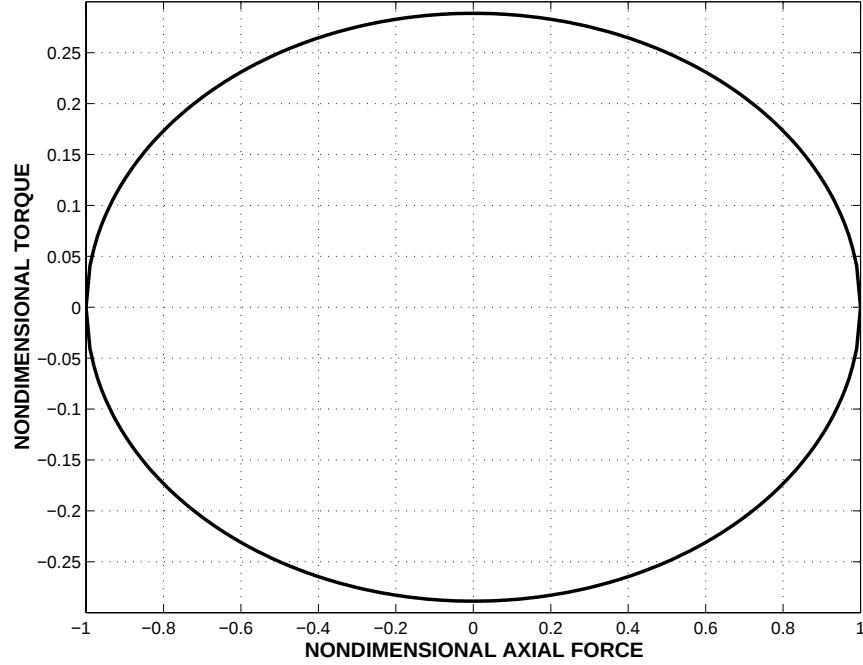


Figure 5.6: Failure criterion in the nondimensional load space. Loading combinations inside the ellipse correspond to safe conditions.

for safe service load conditions. It is convenient to rewrite the criterion in a non dimensional form as

$$\left(\frac{F_1}{\pi R^2 \sigma_{\text{allow}}} \right)^2 + 12 \left(\frac{M_1}{\pi R^3 \sigma_{\text{allow}}} \right)^2 \leq 1. \quad (5.46)$$

Fig. 5.6 shows the geometric interpretation of the criterion: safe loads correspond to combinations inside an ellipse in the nondimensional load space, nondimensional axial force $F_1/(\pi R^2 \sigma_{\text{allow}})$ versus nondimensional torque $M_1/(\pi R^3 \sigma_{\text{allow}})$.

Shaft under torsion and bending

Consider a circular shaft subjected to both bending and torsion, as would occur, for instance, in a cantilever shaft with a tip pulley. Let M_3 and M_1 be the applied bending moment and torque, respectively. The corresponding stresses are

$$\sigma = \frac{2M_3}{\pi R^3}, \text{ and } \tau = \frac{2M_1}{\pi R^3}, \quad (5.47)$$

respectively. Von Mises criterion then requires

$$\left[\left(\frac{2M_3}{\pi R^3} \right)^2 + 3 \left(\frac{2M_1}{\pi R^3} \right)^2 \right]^{1/2} \leq \sigma_{\text{allow}} \quad (5.48)$$

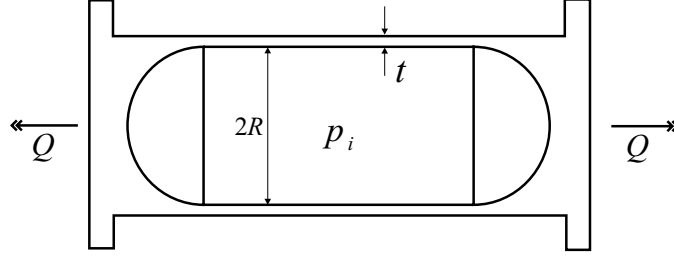


Figure 5.7: Pressure vessel subjected to an external torque.

for safe service load conditions. Here again, these safe load conditions corresponds to the inner portion of an ellipse

$$4 \left(\frac{M_3}{\pi R^3 \sigma_{\text{allow}}} \right)^2 + 12 \left(\frac{M_1}{\pi R^3 \sigma_{\text{allow}}} \right)^2 \leq 1, \quad (5.49)$$

in the nondimensional loading space, nondimensional bending moment $M_3/(\pi R^3 \sigma_{\text{allow}})$ versus nondimensional torque $M_1/(\pi R^3 \sigma_{\text{allow}})$.

5.2.2 Problems

Problem 5.1

Consider the pressure vessel subjected to an internal pressure p_i and an external torque Q . The pressure vessel is of radius R and wall thickness t . Use Von Mises criterion to compute the failure envelope in the space defined by $Q/(tR^2\sigma_{\text{allow}})$ and $p_iR/(t\sigma_{\text{allow}})$.

5.3 Torsion of bars with arbitrary cross-sections

When analyzing the torsional behavior of circular cylinders, the circular symmetry of the problem led to the conclusion that each cross-section rotates about its own center like a rigid disk. If this type of deformation were assumed to remain valid for a bar of arbitrary cross-section, the displacement field (5.3) and (5.4), and the corresponding strain field (5.5) to (5.7) would also describe the kinematics of bars with arbitrary sections. The only non vanishing stress component would be the circumferential shearing stress (5.12).

Unfortunately, this assumption can lead to grossly erroneous results because it implies a solution that violates the equilibrium equations of the problem at the edge of the section. Consider, for instance, the torsion of a rectangular bar as depicted in fig. 5.8. The circumferential shearing stress τ_α given by eq. (5.12) is shown along the edge of the section, and is resolved into its Cartesian components τ_{12} and τ_{13} . The shearing stress component τ_{12} , normal to the edge of the section must clearly vanish since the edges of the bar are stress free. Hence, the only allowable shearing stress component along the edge is τ_{13} , and the stress distribution (5.12) violates equilibrium conditions along the edge of the section.

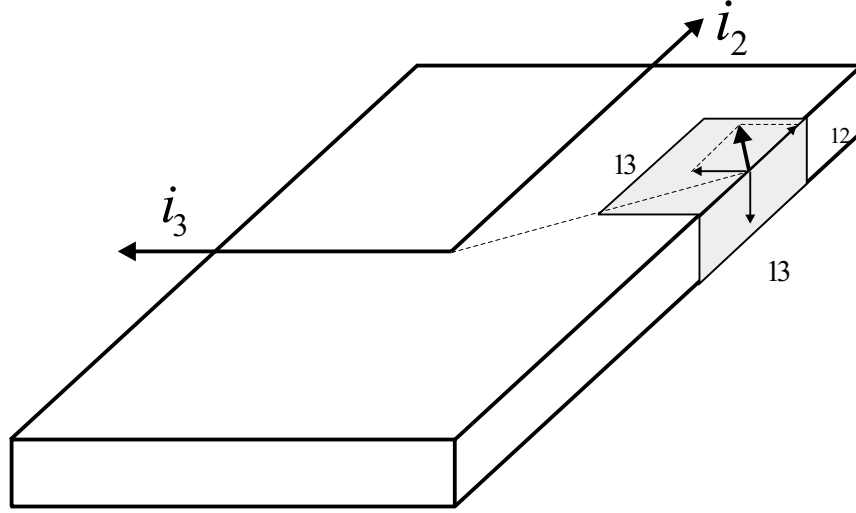


Figure 5.8: Shearing stresses along the edge of a rectangular section.

5.3.1 Saint-Venant's solution

Consider a solid bar with a cross-section of arbitrary shape denoted Ω . A closer look at the problem reveals that for a bar with arbitrary sections, each section rotates like a rigid body, but it is allowed to warp out of its own plane. This type of deformation is described by the following displacement field

$$u_1(x_1, x_2, x_3) = \Psi(x_2, x_3) \kappa_1(x_1); \quad (5.50)$$

$$u_2(x_1, x_2, x_3) = -x_3 \Phi_1(x_1); \quad u_3(x_1, x_2, x_3) = x_2 \Phi_1(x_1). \quad (5.51)$$

The in-plane displacement field (5.51) describes a rigid body rotation of the cross-section, as was the case for the circular cylinder. However, the out-of-plane displacement field does not vanish: it is proportional to the twist rate and has an arbitrary variation over the cross-section described by the unknown warping function $\Psi(x_2, x_3)$. This warping function will be determined by enforcing equilibrium conditions for the resulting shearing stress field. It will be further assumed that the twist rate is constant along the axis of the bar, i.e. $\kappa_1(x_1) = \kappa_1$. This restriction is known as the *uniform torsion* problem.

The strain field associated with the assumed displacement field is

$$\varepsilon_1 = 0; \quad (5.52)$$

$$\varepsilon_2 = 0; \quad \varepsilon_3 = 0; \quad \gamma_{23} = 0; \quad (5.53)$$

$$\gamma_{12} = \left(\frac{\partial \Psi}{\partial x_2} - x_3 \right) \kappa_1; \quad \gamma_{13} = \left(\frac{\partial \Psi}{\partial x_3} + x_2 \right) \kappa_1. \quad (5.54)$$

The vanishing of the axial strain (5.52) is a direct consequence of the uniform torsion assumption, whereas the vanishing of the in-plane strains (5.53) stems from the rigid body

rotation assumption for the in-plane motion of the section. The only non vanishing strain components γ_{12} and γ_{13} depend on the partial derivatives of the unknown warping function.

We now focus on bars made of a linear elastic, isotropic material that obeys Hooke's law (1.91). The stress field is then readily found as

$$\sigma_1 = 0; \quad (5.55)$$

$$\sigma_2 = 0; \quad \sigma_3 = 0; \quad \tau_{23} = 0; \quad (5.56)$$

$$\tau_{12} = G\kappa_1 \left(\frac{\partial \Psi}{\partial x_2} - x_3 \right); \quad \tau_{13} = G\kappa_1 \left(\frac{\partial \Psi}{\partial x_3} + x_2 \right). \quad (5.57)$$

This stress field must satisfy equilibrium eqs. (1.7) at all points of the section. Neglecting body forces, and in view of eq. (5.55), this equilibrium condition reduces to

$$\frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} = 0. \quad (5.58)$$

Hence, the warping function must satisfy the following partial differential equation

$$\frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial^2 \Psi}{\partial x_3^2} = 0. \quad (5.59)$$

at all points of the cross-section Ω .

Furthermore, equilibrium conditions must also be satisfied along the outer edge of the section that defines the contour $\mathcal{C}$. Fig. 5.9 shows a portion of the outer contour $\mathcal{C}$ and the curvilinear variable s that measures length along this contour. The normal component of shearing stress must vanish at all points of $\mathcal{C}$, i.e.

$$\tau_n = 0, \quad (5.60)$$

whereas the component of shearing stress τ_s tangent to the contour does not vanish. In terms of Cartesian components, the normal component of shearing strain is

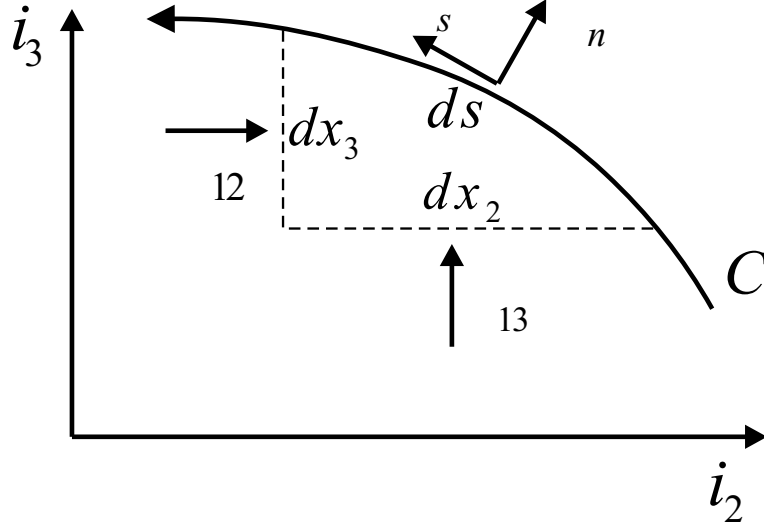
$$\tau_n = \tau_{12} \left(\frac{dx_3}{ds} \right) + \tau_{13} \left(-\frac{dx_2}{ds} \right) = 0. \quad (5.61)$$

Introducing (5.57) then yields the following boundary condition for the warping function

$$\left(\frac{\partial \Psi}{\partial x_2} - x_3 \right) \frac{dx_3}{ds} - \left(\frac{\partial \Psi}{\partial x_3} + x_2 \right) \frac{dx_2}{ds} = 0. \quad (5.62)$$

The warping function which yields a stress field satisfying all equilibrium requirements is the solution of the following partial differential equation and associated boundary condition

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial x_2^2} + \frac{\partial^2 \Psi}{\partial x_3^2} &= 0; & \text{on } \Omega \\ \left(\frac{\partial \Psi}{\partial x_2} - x_3 \right) \frac{dx_3}{ds} - \left(\frac{\partial \Psi}{\partial x_3} + x_2 \right) \frac{dx_2}{ds} &= 0. & \text{along } \mathcal{C} \end{aligned} \quad (5.63)$$

Figure 5.9: Equilibrium condition along the outer contour $\mathcal{C}$.

The solution of this problem is rather complicated in view of the complex boundary condition that must hold along $\mathcal{C}$.

An alternative formulation of the problem that leads to simpler boundary conditions is found by introducing the *stress function* Φ proposed by Prandtl

$$\tau_{12} = \frac{\partial \Phi}{\partial x_3}; \quad \tau_{13} = -\frac{\partial \Phi}{\partial x_2}. \quad (5.64)$$

Shearing stresses derived from this stress function automatically satisfy the equilibrium equation (5.58). Identifying expressions (5.57) and (5.64) for the shearing stresses expressed in terms of warping and stress functions, respectively, yields

$$G\kappa_1 \left(\frac{\partial \Psi}{\partial x_2} - x_3 \right) = \frac{\partial \Phi}{\partial x_3}; \quad G\kappa_1 \left(\frac{\partial \Psi}{\partial x_3} + x_2 \right) = -\frac{\partial \Phi}{\partial x_2}. \quad (5.65)$$

Eliminating the warping function from these expressions then leads to the following partial differential equation for the stress function

$$\frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} = -2G\kappa_1. \quad (5.66)$$

The boundary conditions along $\mathcal{C}$ then follow from eqs. (5.61) and (5.64)

$$\tau_n = \frac{\partial \Phi}{\partial x_3} \frac{dx_3}{ds} + \frac{\partial \Phi}{\partial x_2} \frac{dx_2}{ds} = \frac{d\Phi}{ds} = 0. \quad (5.67)$$

which implies a constant value of Φ along the contour $\mathcal{C}$. If the section is bounded by several disconnected curves, the stress function must be a constant along each individual curve,

though the value of the constant is different for each curve. For solid cross-sections bounded by a single curve, the constant value of the stress function along that curve may be chosen as zero, as this choice has no effect on the resulting stress distribution. The stress function is the solution of the following partial differential equation and associated boundary condition

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} &= -2G\kappa_1. & \text{on } \Omega \\ \frac{d\Phi}{ds} &= 0 & \text{along } \mathcal{C}. \end{aligned} \quad (5.68)$$

The differential equations for the warping and stress functions were found from local equilibrium consideration. Global equilibrium of the section must also be verified. For a solid section bounded by a single contour, the resultant shearing forces acting on the section are

$$F_2 = \int_{\Omega} \tau_{12} d\Omega = \int_{x_2} \int_{x_3} \frac{\partial \Phi}{\partial x_3} dx_2 dx_3 = \int_{x_2} \left[\int_{x_3} \frac{\partial \Phi}{\partial x_3} dx_3 \right] dx_2 = 0, \quad (5.69)$$

and

$$F_3 = \int_{\Omega} \tau_{13} d\Omega = \int_{x_2} \int_{x_3} -\frac{\partial \Phi}{\partial x_2} dx_2 dx_3 = - \int_{x_3} \left[\int_{x_2} \frac{\partial \Phi}{\partial x_2} dx_2 \right] dx_3 = 0; \quad (5.70)$$

where the last equalities follow from selecting a zero value for the stress function along the contour $\mathcal{C}$. The total torque acting on the section is

$$M_1 = \int_{\Omega} (x_2 \tau_{13} - x_3 \tau_{12}) d\Omega = \int_{\Omega} \left(-x_2 \frac{\partial \Phi}{\partial x_2} - x_3 \frac{\partial \Phi}{\partial x_3} \right) d\Omega. \quad (5.71)$$

Integrating by parts and taking advantage of the vanishing of the stress function along $\mathcal{C}$ yields

$$M_1 = 2 \int_{\Omega} \Phi d\Omega. \quad (5.72)$$

In summary, the stress distribution in a bar of arbitrary cross-section subjected to uniform torsion can be obtained by evaluating either the warping or stress function from (5.63) or (5.68), respectively. The stress field then follows from eqs. (5.57) or (5.64), respectively. This represents an exact solution of the problem as all governing equations are satisfied.

Example: Torsion of an elliptical bar

Consider a bar with an elliptical cross-section as shown in fig. 5.10. The equation for the contour $\mathcal{C}$ defining the section is

$$\left(\frac{x_2}{a} \right)^2 + \left(\frac{x_3}{b} \right)^2 = 1.$$

A stress function of the following form is assumed

$$\Phi = A \left[\left(\frac{x_2}{a} \right)^2 + \left(\frac{x_3}{b} \right)^2 - 1 \right],$$

where A is an unknown constant. The boundary condition, eq. (5.67), is clearly satisfied since $\Phi = 0$ along $\mathcal{C}$. The governing differential eq. (5.66) is also satisfied for the following value of A

$$A \left(\frac{2}{a^2} + \frac{2}{b^2} \right) = -2G\kappa_1; \quad \implies \quad A = -\frac{a^2b^2}{a^2 + b^2} G\kappa_1.$$

The stress function then becomes

$$\Phi = -\frac{a^2b^2}{a^2 + b^2} \left[\left(\frac{x_2}{a} \right)^2 + \left(\frac{x_3}{b} \right)^2 - 1 \right] G\kappa_1.$$

The torque now follows from (5.72)

$$M_1 = -\frac{2a^2b^2}{a^2 + b^2} G\kappa_1 \int_{\Omega} \left[\left(\frac{x_2}{a} \right)^2 + \left(\frac{x_3}{b} \right)^2 - 1 \right] d\Omega.$$

Note that

$$\int_{\Omega} d\Omega = \pi ab; \quad \int_{\Omega} x_2^2 d\Omega = \frac{\pi a^3 b}{4}; \quad \int_{\Omega} x_3^2 d\Omega = \frac{\pi ab^3}{4};$$

are the area, and the second moments of area about $\mathbf{i}_2$ and $\mathbf{i}_3$, respectively, for the ellipse. The torque then becomes

$$M_1 = G \frac{\pi a^3 b^3}{a^2 + b^2} \kappa_1 = J \kappa_1,$$

where the torsional stiffness of the elliptical section is found as

$$J = G \frac{\pi a^3 b^3}{a^2 + b^2}. \quad (5.73)$$

The stress function is readily expressed in terms of the applied torque

$$\Phi = -\frac{1}{\pi ab} \left[\left(\frac{x_2}{a} \right)^2 + \left(\frac{x_3}{b} \right)^2 - 1 \right] M_1.$$

Finally, the stress distribution is found from eq. (5.64)

$$\tau_{12} = -\frac{2x_3}{\pi ab^3} M_1; \quad \tau_{13} = \frac{2x_2}{\pi a^3 b} M_1.$$

The maximum shear stress occurs at the boundary of the section. At points A and B the stresses are

$$\tau_{12}^B = -\frac{2M_1}{\pi ab^2}; \quad \tau_{13}^A = \frac{2M_1}{\pi a^2 b}.$$

Clearly, the maximum shear stress occurs at the end of the minor axis of the ellipse, i.e. at point B where

$$|\tau_{max}| = \frac{2M_1}{\pi ab^2}.$$

The warping function is readily obtained from eq. (5.65)

$$\Psi = -\frac{a^2 - b^2}{a^2 + b^2} x_2 x_3.$$

For $a = b = R$ the bar with an elliptical section becomes a circular cylinder of radius R . The torsional stiffness for the elliptical section reduces to (5.21), and the maximum shear stress to (5.27). Finally, the warping function vanishes as was determined by symmetry arguments for the circular cylinder.

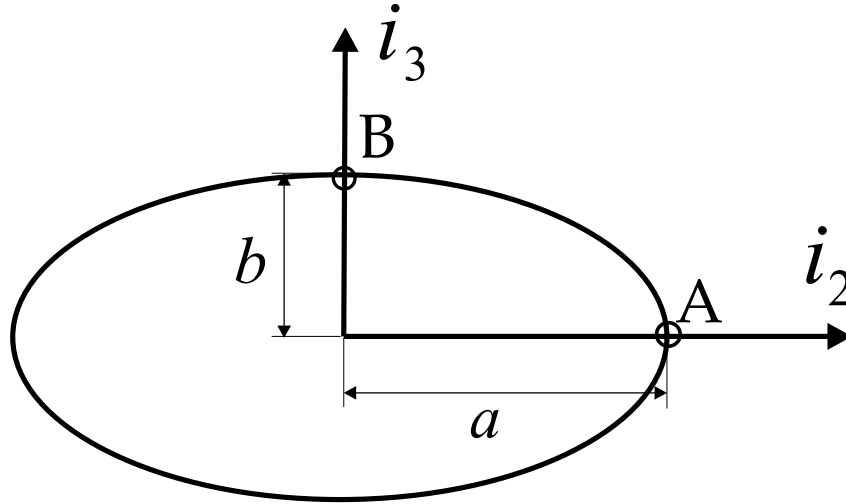


Figure 5.10: A bar with an elliptical cross-section.

5.3.2 Problems

Problem 5.2

Consider a circular tube of inner radius R_i and outer radius R_o made of a homogeneous, isotropic material of shearing modulus G , as shown in fig. 5.11.

1. Find the Airy stress function for this problem starting from

$$\Phi = Ar^2,$$

where $r^2 = x_2^2 + x_3^2$.

2. Is your solution the exact solution of the problem?
3. Evaluate the torsional stiffness of the tube based on the Airy stress function.
4. Plot the distribution of shearing stresses. Find the location and magnitude of the maximum shearing stress.
5. Simplify your results for the case of a thin-walled tube, $R_m = (R_i + R_o)/2$, $t = (R_o - R_i) \ll R_m$. How do the predictions agree with those of thin-wall theory?

Problem 5.3

Consider a circular shaft of radius a with a semi-circular keyway of radius b , as depicted in fig. 5.12. The shaft is subjected to torsion.

1. Find the Airy stress function for this problem starting from

$$\Phi = A(x_2^2 + x_3^2 - 2ax_2) \left(1 - \frac{b^2}{x_2^2 + x_3^2}\right).$$

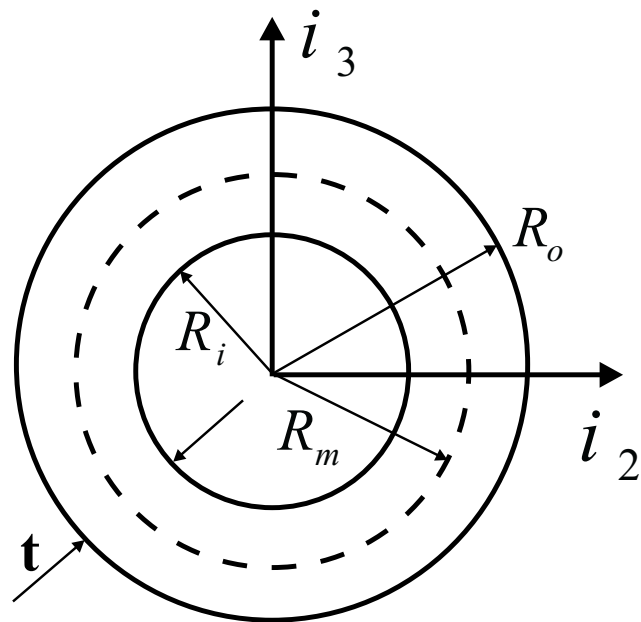


Figure 5.11: Cross-section of a circular tube.

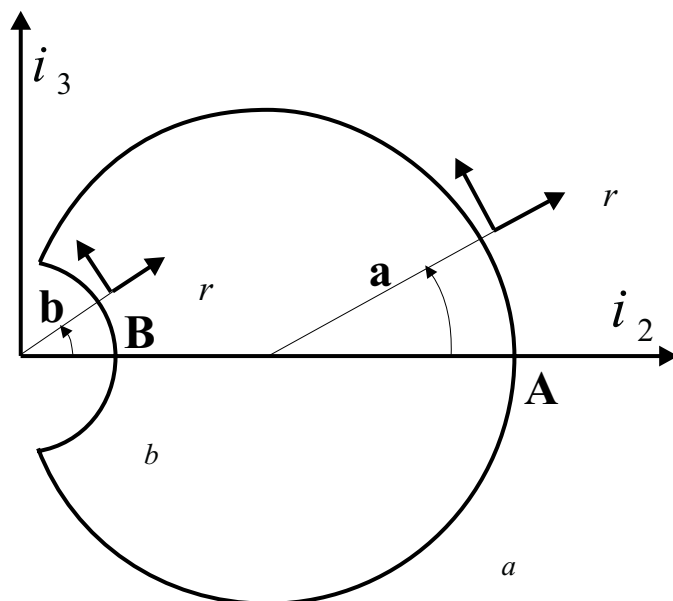


Figure 5.12: Circular shaft with a keyway.

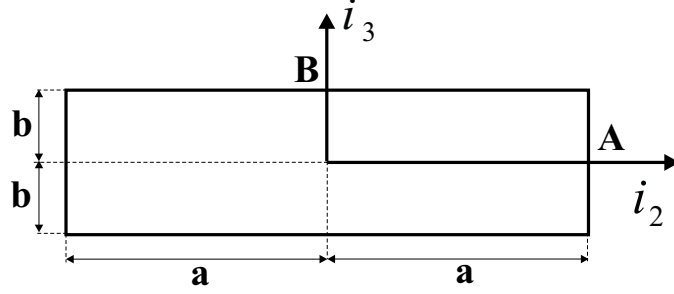


Figure 5.13: Bar with a rectangular cross-section.

2. Find the shear stress distribution $\tau_r = \tau_r(\alpha)$ and $\tau_\alpha = \tau_\alpha(\alpha)$ along the contour Γ_a of the shaft.
3. Find the shear stress distribution $\tau_r = \tau_r(\beta)$ and $\tau_\beta = \tau_\beta(\beta)$ along the contour Γ_b of the keyway.
4. Let $\tau_N = G\kappa_1 a$ be the shaft maximum shearing stress *in the absence of keyway*. Find

$$\lim_{b \rightarrow 0} \frac{\tau_\alpha^A}{\tau_N}; \quad \lim_{b \rightarrow 0} \frac{\tau_\beta^B}{\tau_N}.$$

Comment on your results.

Problem 5.4

The narrow rectangular strip ($a \gg b$) shown in fig. 5.13 is the cross-section of a bar subjected to torsion. Investigate the behavior of this section under torsion using an energy approach. Use a stress function of the following type

$$\Phi = G\kappa_1 (b^2 - x_3^2) \left(1 - e^{-\alpha(a - |x_2|)}\right),$$

where α is an unknown parameter. Assume $e^{-\alpha a} \approx 0$.

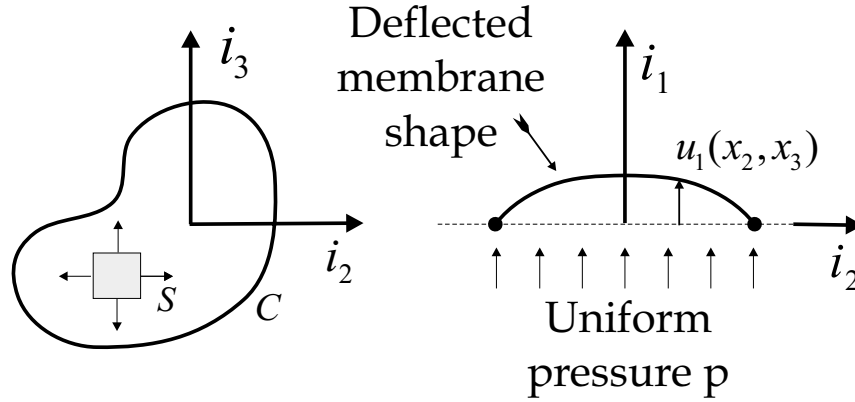
1. Find the torsional stiffness of the bar.
2. Find the shearing stresses at points A and B

$$\frac{(2a)(2b)^2 \tau_A}{M_1}, \quad \frac{(2a)(2b)^2 \tau_B}{M_1},$$

where M_1 is the applied torque.

5.4 The membrane analogy

Saint-Venant's solution expresses the shearing stress distribution over the cross-section of a bar under torsion in terms of the warping or stress functions which satisfy the partial differential eqs. (5.63) or (5.68), respectively. Though this procedure completely solves the

Figure 5.14: The thin membrane attached to the contour $\mathcal{C}$.

problem, the analyst has little intuition about the shape of the warping or stress functions for a given cross-sectional shape.

Prandtl developed an analogy between the shape of the stress function and the deflected shape of a thin, uniformly stretched membrane subjected to a uniform transverse pressure. Since the solution of this latter problem is rather intuitive, it provides valuable insight about the shape of the stress function.

Consider a thin, uniformly stretched membrane attached to a planar curve $\mathcal{C}$ defining a domain Ω , and subjected to a uniform transverse pressure p , as depicted in fig. 5.14. Under the effect of the pressure, the membrane deflects upwards and reaches an equilibrium shape $u_1(x_2, x_3)$ where the applied pressure is equilibrated by the out-of-plane component of the uniform stretching force S .

The deflected shape of the membrane is evaluated by considering the differential element shown in fig. 5.15. The force $p \, dx_2 dx_3$ due to the applied pressure is equilibrated by the out-of-plane components of the stretching forces

$$\begin{aligned} p \, dx_2 dx_3 - S \frac{\partial u_1}{\partial x_2} \, dx_3 + S \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2} \, dx_2 \right) \, dx_3 \\ - S \frac{\partial u_1}{\partial x_3} \, dx_2 + S \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial^2 u_1}{\partial x_3^2} \, dx_3 \right) \, dx_2 = 0. \end{aligned} \quad (5.74)$$

After simplification, this equilibrium condition that must hold at all points of Ω becomes

$$\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} = -\frac{p}{S} \quad \text{on } \Omega. \quad (5.75)$$

Along the planar curve $\mathcal{C}$, the deflected shape of the membrane must remain constant, i.e.

$$\frac{du_1}{ds} = 0 \quad \text{along } \mathcal{C}. \quad (5.76)$$

The deflected shape of the membrane is found by solving the partial differential equation (5.75) subjected to the boundary condition (5.76).

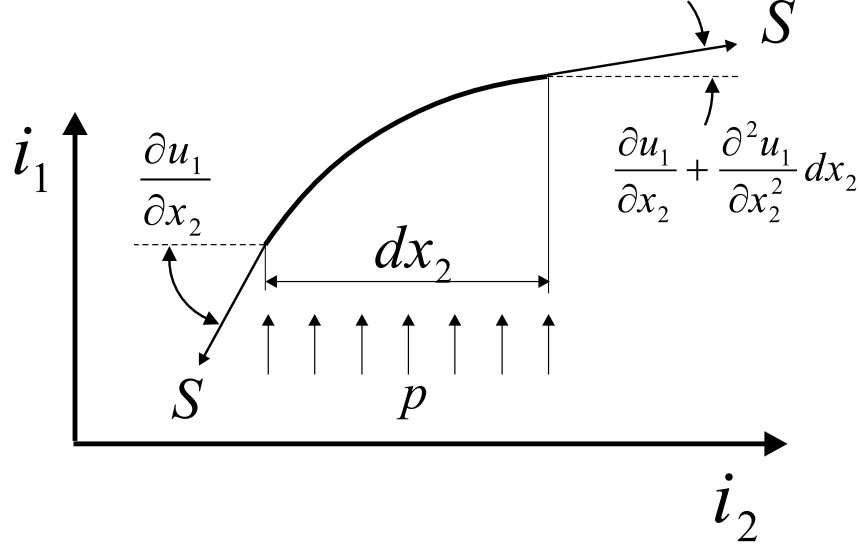


Figure 5.15: A differential element of the membrane.

The membrane analogy now follows from comparing the governing equations and boundary conditions (5.68) for the stress function with the corresponding eqs. (5.75) and (5.76) for the deflection of the membrane. Clearly, the two sets of equations are identical if

$$\frac{\Phi}{2G\kappa_1} = \frac{u_1}{p/S}. \quad (5.77)$$

This analogy implies that the stress function corresponding to the torsional problem of a bar of cross-sectional area Ω with an outer contour $\mathcal{C}$ is equal to the deflected shape of a membrane stretched over the same contour, within a normalizing constant $(2G\kappa_1)/(p/S)$.

This analogy is further extended by considering contours of constant deflection of the membrane. Fig. 5.16 shows such a contour denoted $\bar{\mathcal{C}}$ that encloses an area $\mathcal{A}$. Let s be the curvilinear variable measuring length along $\bar{\mathcal{C}}$, and $\mathbf{s}$ and $\mathbf{n}$ unit vectors respectively tangent and normal to $\bar{\mathcal{C}}$. It is clear that $\partial u_1 / \partial s = 0$ since $\bar{\mathcal{C}}$ is a contour of constant membrane deflection, and $\partial u_1 / \partial n$ represents the slope of the membrane in the direction normal to $\bar{\mathcal{C}}$. On the other hand, the components of shearing stress τ_n and τ_s in the direction normal and tangent to $\bar{\mathcal{C}}$ are

$$\tau_n = \frac{\partial \Phi}{\partial s}; \quad \tau_s = -\frac{\partial \Phi}{\partial n}, \quad (5.78)$$

where n measures length in the direction normal to $\bar{\mathcal{C}}$. Introducing (5.77) then yields

$$\tau_n = \frac{2G\kappa_1}{p/S} \frac{\partial u_1}{\partial s} = 0; \quad \tau_s = -\frac{2G\kappa_1}{p/S} \frac{\partial u_1}{\partial n}. \quad (5.79)$$

This means that at any point, the shear stress is tangent to the contour of constant membrane deflection passing through that point, and the magnitude of the shearing stress is proportional to the slope of the membrane in the direction normal to that contour. This

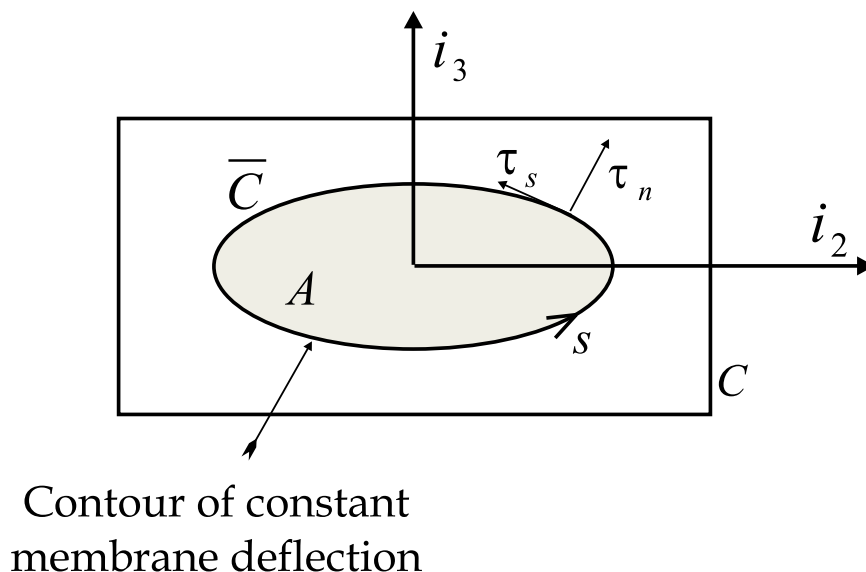


Figure 5.16: Contours of constant membrane deflection.

conclusion also applies to $\mathcal{C}$ the outer contour of the cross-section where the shearing stress is tangent to $\mathcal{C}$ as required by the local equilibrium condition (5.60), and its magnitude is proportional to the slope of the membrane in the direction normal to the outer contour.

Consider now the equilibrium of the portion $\mathcal{A}$ of the membrane enclosed by $\bar{\mathcal{C}}$. The force $p\mathcal{A}$ due to the applied pressure is equilibrated by the normal component of the stretching force

$$p\mathcal{A} + \int_{\bar{\mathcal{C}}} S \frac{\partial u_1}{\partial n} ds = 0. \quad (5.80)$$

We then introduce the analogy (5.77) and simplify to find

$$\frac{1}{\bar{l}} \int_{\bar{\mathcal{C}}} \tau_s ds = 2G\kappa_1 \frac{\mathcal{A}}{\bar{l}}. \quad (5.81)$$

where $\bar{l}$ is the length of the closed contour $\bar{\mathcal{C}}$. In other words, the average shearing stress along a contour of constant membrane deflection is proportional to $\mathcal{A}/\bar{l}$, i.e. the ratio of the area enclosed by $\bar{\mathcal{C}}$ to its perimeter.

Finally, the torque given by (5.72) is related to the volume V under the deflected membrane

$$M_1 = 2 \int_{\Omega} d\Omega = \frac{2G\kappa_1}{p/S} 2 \int_{\Omega} d\Omega = \frac{2G\kappa_1}{p/S} 2V. \quad (5.82)$$

5.5 Torsion of a thin rectangular section

The torsional behavior of a thin rectangular section will be investigated with the help of the membrane analogy. Consider the thin rectangular strip shown in fig. 5.17, where $t \ll b$. The membrane stretched over this thin strip will deflect into a cylindrical shape with no appreciable slope in the i_3 direction, except near the far edges of the section. This means

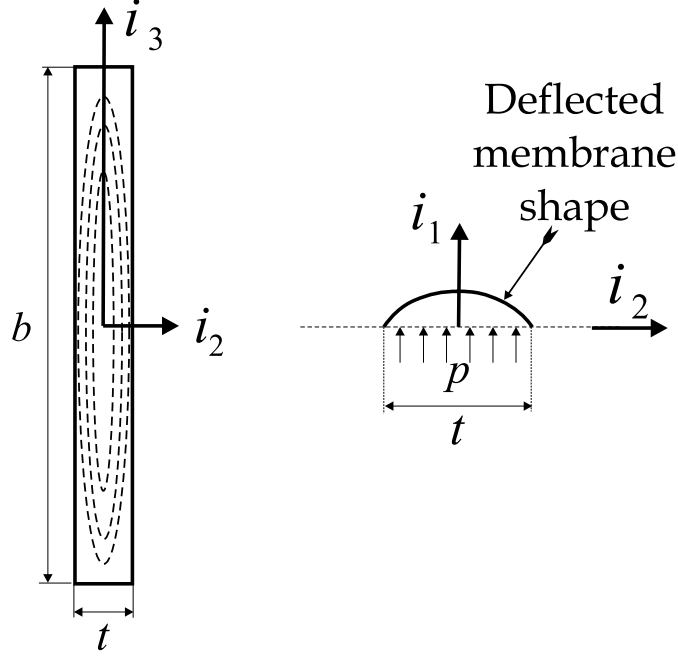


Figure 5.17: A thin rectangular strip.

that it is reasonable to assume $\partial u_1 / \partial x_3 \approx 0$, and the governing equation for the membrane (5.75) becomes

$$\frac{\partial^2 u_1}{\partial x_2^2} = -\frac{p}{S}. \quad (5.83)$$

Integrating and enforcing $u_1 = 0$ at the edges of the contour, i.e. at $x_2 = \pm t/2$ then yields

$$u_1 = \frac{1}{2} \left(\frac{t}{2} \right)^2 \left[1 - \left(\frac{2x_2}{t} \right)^2 \right] \frac{p}{S}. \quad (5.84)$$

The membrane analogy (5.77) then gives the corresponding stress function

$$\Phi = \left(\frac{t}{2} \right)^2 \left[1 - \left(\frac{2x_2}{t} \right)^2 \right] G\kappa_1. \quad (5.85)$$

The stress distribution now follows from eq. (5.64)

$$\tau_{12} \approx 0; \quad \tau_{13} = 2x_2 G\kappa_1, \quad (5.86)$$

and the torque associated with this stress distribution is evaluated with the help of (5.72)

$$M_1 = 2G\kappa_1 \left(\frac{t}{2} \right)^2 b \int_{-t/2}^{+t/2} \left[1 - \left(\frac{2x_2}{t} \right)^2 \right] dx_2 = \frac{bt^3}{3} G\kappa_1. \quad (5.87)$$

Here again a linear relationship is found between the applied torque and the resulting twist rate. The torsional stiffness for the rectangular strip is now

$$J = G \frac{bt^3}{3}. \quad (5.88)$$

The torsional stiffness is proportional to t^3 , i.e. a thin rectangular strip has a very low torsional stiffness. Eliminating the twist rate between eqs. (5.86) and (5.87) yields the stress distribution in terms of the applied torque

$$\tau_{12} \approx 0; \quad \tau_{13} = \frac{3M_1}{bt^2} \left(\frac{2x_2}{t} \right). \quad (5.89)$$

This corresponds to a distribution of shear stress that is linear through the thickness of the thin rectangular strip, as depicted in fig. 5.18. The shearing stress vanishes at the wall mid-line and reaches its maximum value along the two edges of the section. Clearly,

$$\tau_{max} = \frac{3M_1}{bt^2}. \quad (5.90)$$

The following observation gives interesting insight into the torsional behavior of thin rectangular strips. The torque resulting from the stress distribution (5.89) is

$$\int_0^b \int_{-t/2}^{+t/2} \tau_{13} x_2 dx_2 dx_3 = \left[\int_0^b dx_3 \right] \left[\frac{6M_1}{bt^3} \int_{-t/2}^{+t/2} x_2^2 dx_2 \right] = \frac{M_1}{2}. \quad (5.91)$$

This means that the shearing stresses τ_{13} only account for half of the torque. The other half must be associated with τ_{12} . Though this stress component is very small, its contribution to the torque is multiplied by a moment arm $b/2$, as compared to the moment arm $t/2$ for the τ_{13} component.

5.6 Torsion of thin-walled open sections

The results of the previous example are readily extended to thin-walled open sections. For thin walls, the deflected shape of the membrane is once again expected to be cylindrical, except near the edges of the section. The developments of the previous example still apply and the torsional stiffness of a thin walled open section become

$$J = G \frac{lt^3}{3}, \quad (5.92)$$

where l is the length of the contour and t the thickness. For eq. (5.86), the maximum shear stress occurs along the walls of the section

$$\tau_s^{max} = Gt \kappa_1. \quad (5.93)$$

The maximum shearing stress expressed in terms of the applied torque becomes

$$\tau_s^{max} = \frac{3M_1}{lt^2}. \quad (5.94)$$

Consider, as an example, the C-channel shown in fig. 5.19. The torsional stiffness is

$$J = G \left[\frac{bt_f^3}{3} + \frac{ht_w^3}{3} + \frac{bt_f^3}{3} \right] = G \left[\frac{ht_w^3}{3} + \frac{2bt_f^3}{3} \right], \quad (5.95)$$

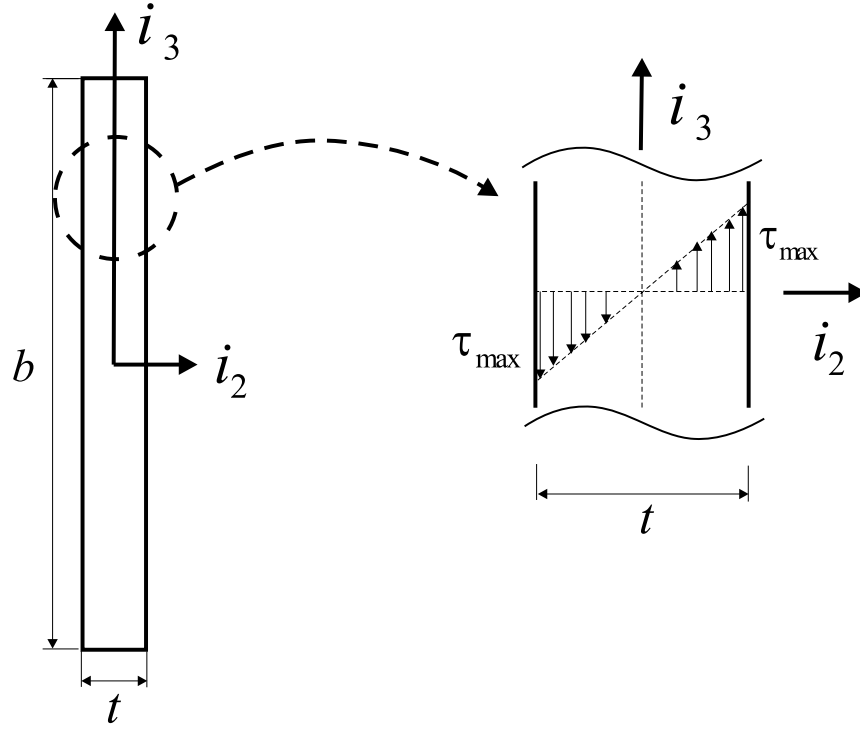


Figure 5.18: Shear stress distribution through the thickness of a thin rectangular strip.

and the maximum shear stress occurs along the walls of the section

$$\tau_s^{max} = Gt\kappa_1. \quad (5.96)$$

The shearing stresses in the web and flange, denoted τ_w and τ_f , respectively, become

$$\tau_w = Gt_w\kappa_1 = \frac{Gt_w}{J} M_1; \quad \tau_f = Gt_f\kappa_1 = \frac{Gt_f}{J} M_1. \quad (5.97)$$

5.6.1 Problems

Problem 5.5

Fig. 5.20 depicts the thin-walled, semi-circular open cross-section of a beam. The wall thickness is t , and the material Young's and shearing moduli are E and G , respectively.

1. Find the distribution of shear stress due to an applied torque Q .
2. Indicate the location and magnitude of the maximum shear stress $Rt^2\tau_{max}/Q$.

Problem 5.6

Fig. 5.21 depicts the cross-section of a thin-walled beam.

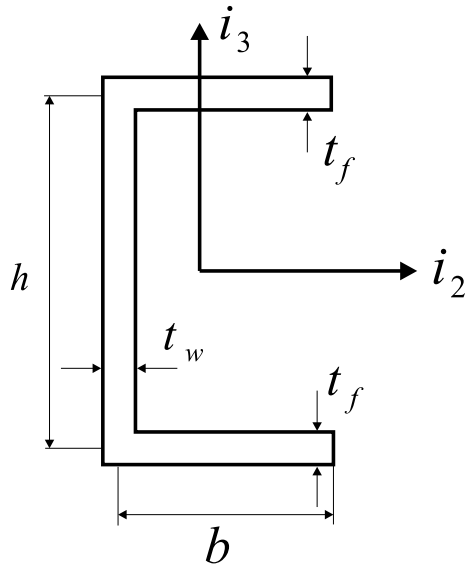


Figure 5.19: A thin-walled C-channel section

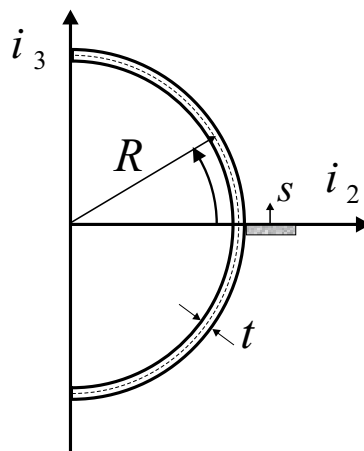


Figure 5.20: Semi-circular open cross-section.

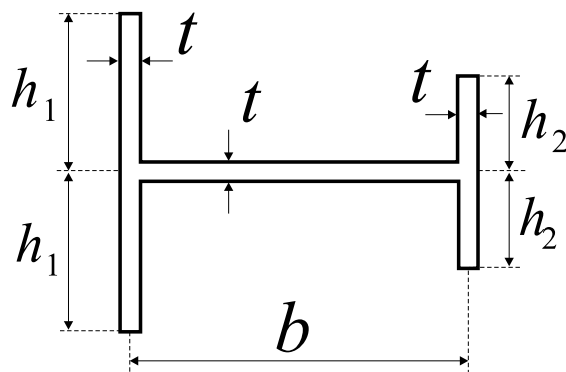


Figure 5.21: Cross-section of a thin-walled beam.

1. Find the magnitude and location of the maximum shearing stress if the section is subjected to a torque Q .
2. Sketch the distribution of shear stress through the thickness of the wall.

Chapter 6

Thin-Walled Beams.

Typical aeronautical structures involve light-weight, thin-walled beam-like structures that must operate in a complex loading environment where axial, bending, shearing, and torsional loads are present. These structures can present closed or open sections, or a combination of both. A closed cross-section is one for which the thin wall forms a closed path. It is an open section in the opposite case. This distinction has profound implications on the structural response of the beam, most importantly when it comes to shearing and torsion.

6.1 Basic equations for thin-walled beams.

In the analysis of thin-walled beams, the specific geometric nature of the beam consisting of an assembly of thin sheets, will be exploited to simplify the problem formulation. Figs. 6.1 to 6.4 show different types of thin-walled cross-sections. Fig. 6.1 shows a beam with a closed section, as opposed to the open section of fig. 6.2. A combination of both types, depicted in fig. 6.3 is also possible. Finally, multi-cellular sections such as shown in fig. 6.4 are very common in aeronautical constructions.

The geometry of the section is described by a curve $\mathcal{C}$ drawn along the mid-thickness of the wall. A curvilinear variable s measuring length along this contour is defined with an arbitrary origin. This variable defines an orientation along $\mathcal{C}$ at all points. Of course, this orientation can be chosen arbitrarily. The wall thickness $t(s)$ can vary from point to point along the contour. For multi-cellular sections, a number of different curves are used to completely describe the section, and a corresponding number of curvilinear variables define the length and orientation of these various curves.

6.1.1 The thin wall assumption

In thin-walled beams the wall thickness is assumed to be much smaller than a representative dimension of the cross-section. Considering fig. 6.1, this means

$$\frac{t(s)}{b} \ll 1. \quad (6.1)$$

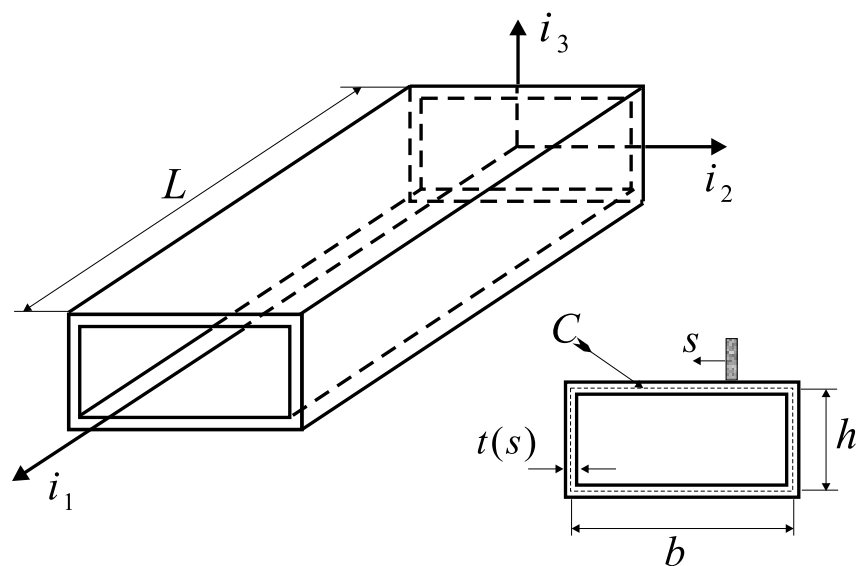


Figure 6.1: Thin-walled beam with a closed, single cell section.

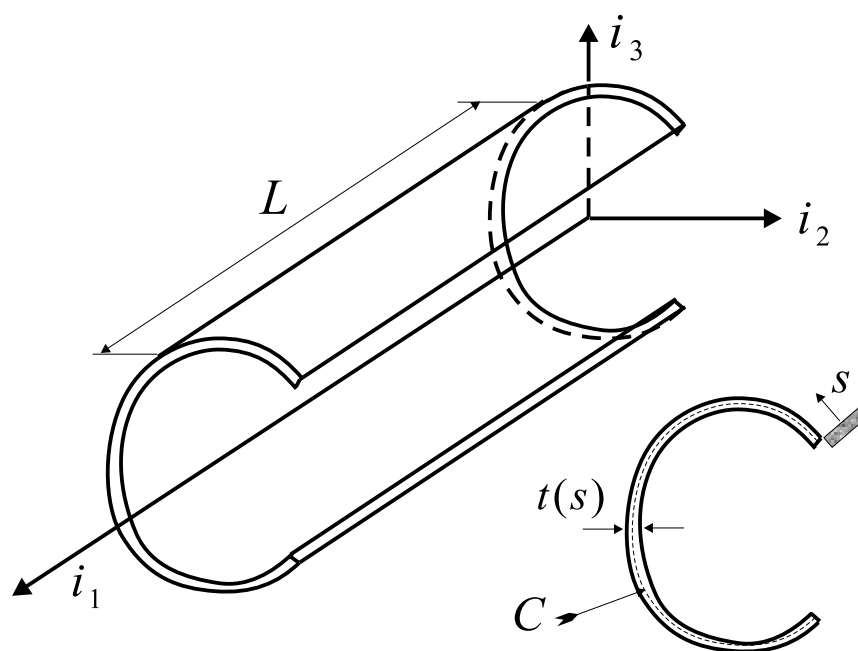


Figure 6.2: Thin-walled beam with an open section.

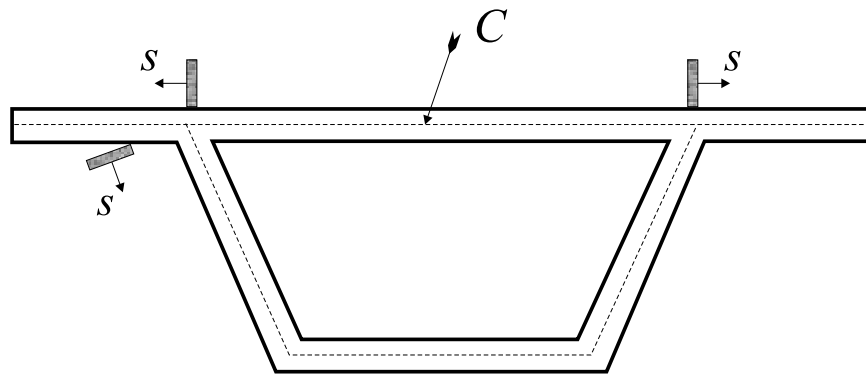


Figure 6.3: Thin-walled beam with open and closed components.

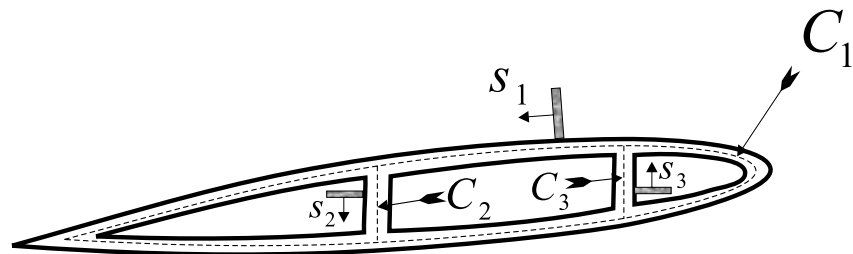


Figure 6.4: Thin-walled beam with a multi-cellular section.

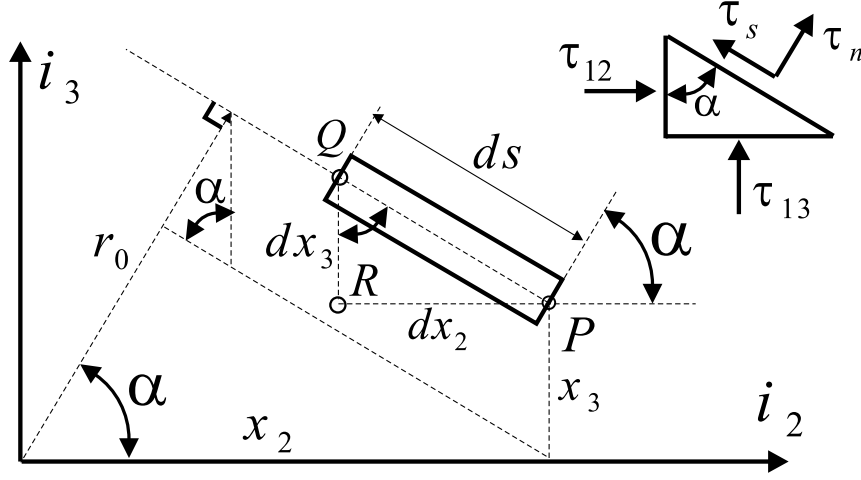


Figure 6.5: Geometry of a differential element of the wall.

Of course, for beam theory to be a reasonable approximation to the structural behavior, the thin-walled beam must be long, i.e.

$$\frac{b}{L} \ll 1. \quad (6.2)$$

6.1.2 Stress flows

The first approximation of thin-walled beam theory concerns the distribution of stress components across the wall thickness. Beam theory assumes the cross-section to be infinitely rigid in its own plane, and as a result, the stress components σ_2 , σ_3 , and τ_{23} , are negligible. The only non vanishing stress component are the axial stress σ_1 and the shearing stresses τ_{12} and τ_{13} . It is convenient to resolve the shearing stress into components τ_s and τ_n , respectively parallel and normal to $\mathcal{C}$. The relationship between these two sets of stresses is readily found by inspection of fig. 6.5

$$\tau_n = \cos \alpha \tau_{12} + \sin \alpha \tau_{13} = \tau_{12} \frac{dx_3}{ds} - \tau_{13} \frac{dx_2}{ds}; \quad (6.3)$$

$$\tau_s = -\sin \alpha \tau_{12} + \cos \alpha \tau_{13} = \tau_{12} \frac{dx_2}{ds} + \tau_{13} \frac{dx_3}{ds}, \quad (6.4)$$

where basic trigonometric relationships applied to triangle PQR were used to show that $\cos \alpha = dx_3/ds$ and $\sin \alpha = -dx_2/ds$.

The shearing stress component τ_n should vanish at the two edges of the wall because the outer surfaces of the beam are stress free. Furthermore, since the wall is very thin, no appreciable magnitude of shearing stress τ_n can build up within the structure, i.e. $\tau_n \approx 0$ through the wall thickness. The only non vanishing shearing stress component is τ_s , taken positive in the direction of s . Inverting relations (6.3) and (6.4) then yields

$$\tau_{12} = \cos \alpha \tau_n - \sin \alpha \tau_s = \tau_s \frac{dx_2}{ds}; \quad (6.5)$$

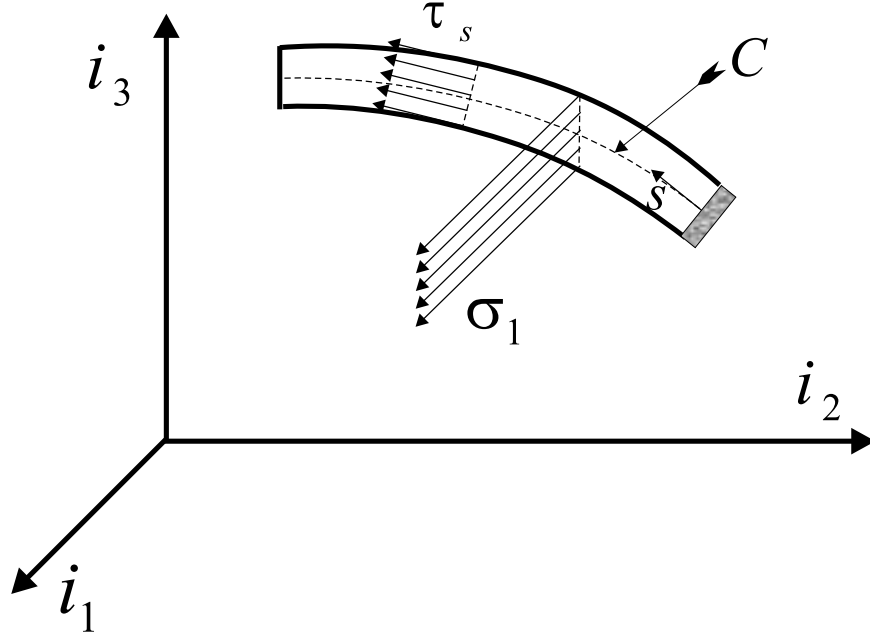


Figure 6.6: Uniform distributions of axial and shear stresses across the wall thickness.

$$\tau_{13} = \sin \alpha \tau_n + \cos \alpha \tau_s = \tau_s \frac{dx_3}{ds}. \quad (6.6)$$

Since the wall is very thin, it seems reasonable to assume that the non vanishing stress components are uniformly distributed across the wall thickness. Fig. 6.6 shows the axial and shearing stress components through the thickness of the wall. It is customary to introduce the concept of *stress flows* defined as

$$n(x_1, s) = \sigma_1(x_1, s) t(s); \quad f(x_1, s) = \tau_s(x_1, s) t(s), \quad (6.7)$$

where n is the *axial stress flow* or *axial flow*, and f the *shearing stress flow* or *shear flow* taken positive in the direction of s .

6.1.3 Stress resultants

The definitions of stress resultant in thin-walled beams are identical to those given in section (3.3) for beams with solid sections. Due to the thin wall assumption, the integration over the cross-sectional area of the beam can be viewed as an integration along $\mathcal{C}$. For instance, the axial force is

$$F_1(x_1) = \int_{\Omega} \sigma_1 d\Omega = \int_{\mathcal{C}} \sigma_1 t ds = \int_{\mathcal{C}} n ds, \quad (6.8)$$

where the definition of the axial flow was used. The bending moments are similarly found

$$M_2(x_1) = \int_{\mathcal{C}} n x_3 ds; \quad M_3(x_1) = - \int_{\mathcal{C}} n x_2 ds. \quad (6.9)$$

The resultant shear forces along axes $\mathbf{i}_2$ and $\mathbf{i}_3$ are

$$F_2(x_1) = \int_{\Omega} \tau_{12} d\Omega = \int_{\mathcal{C}} f \frac{dx_2}{ds} ds; \quad F_3(x_1) = \int_{\Omega} \tau_{13} d\Omega = \int_{\mathcal{C}} f \frac{dx_3}{ds} ds, \quad (6.10)$$

where relationships (6.5) and (6.6) were used. Finally, the resultant torque about the origin of the axes O is

$$M_{O1}(x_1) = \int_{\mathcal{C}} f r_o ds, \quad (6.11)$$

where r_o is the distance from O to the tangent to the contour at a point P . An inspection of fig. 6.5 yields

$$r_o = x_2 \cos \alpha + x_3 \sin \alpha = x_2 \frac{dx_3}{ds} - x_3 \frac{dx_2}{ds}. \quad (6.12)$$

It will be necessary to evaluate the torque about other points of the section, say point K of coordinates (x_{2k}, x_{3k}) ,

$$M_{1K}(x_1) = \int_{\mathcal{C}} f r_k ds, \quad (6.13)$$

where

$$r_k = (x_2 - x_{2k}) \frac{dx_3}{ds} - (x_3 - x_{3k}) \frac{dx_2}{ds} = r_o - x_{2k} \frac{dx_3}{ds} + x_{3k} \frac{dx_2}{ds}. \quad (6.14)$$

6.1.4 Local equilibrium equation

Consider now the differential element of wall shown in fig. 6.7. The dimensions of the differential element are dx_1 along the axis of the beam, and ds along $\mathcal{C}$. Summing up all the forces acting on this element in the axial direction $\mathbf{i}_1$ yields

$$-n ds + \left(n + \frac{\partial n}{\partial x_1} dx_1 \right) ds - f dx_1 + \left(f + \frac{\partial f}{\partial s} ds \right) dx_1 = 0, \quad (6.15)$$

where body forces were neglected. After simplification, this equilibrium condition becomes

$$\frac{\partial n}{\partial x_1} + \frac{\partial f}{\partial s} = 0. \quad (6.16)$$

This local equilibrium equation implies that any change in axial flow along the beam axis must be equilibrated by a corresponding change in shear flow across the section.

6.1.5 Evaluation of sectional stiffnesses

The thin wall assumption also simplifies the evaluation of the bending stiffnesses of the section. Consider the homogeneous, thin-walled rectangular box beam shown in fig. 6.8. The inner and outer heights are h_i and h_o , respectively, whereas the inner and outer width are b_i and b_o , respectively. The thickness of the flange t_f and web t_w are then

$$t_f = \frac{h_o - h_i}{2}; \quad t_w = \frac{b_o - b_i}{2}. \quad (6.17)$$

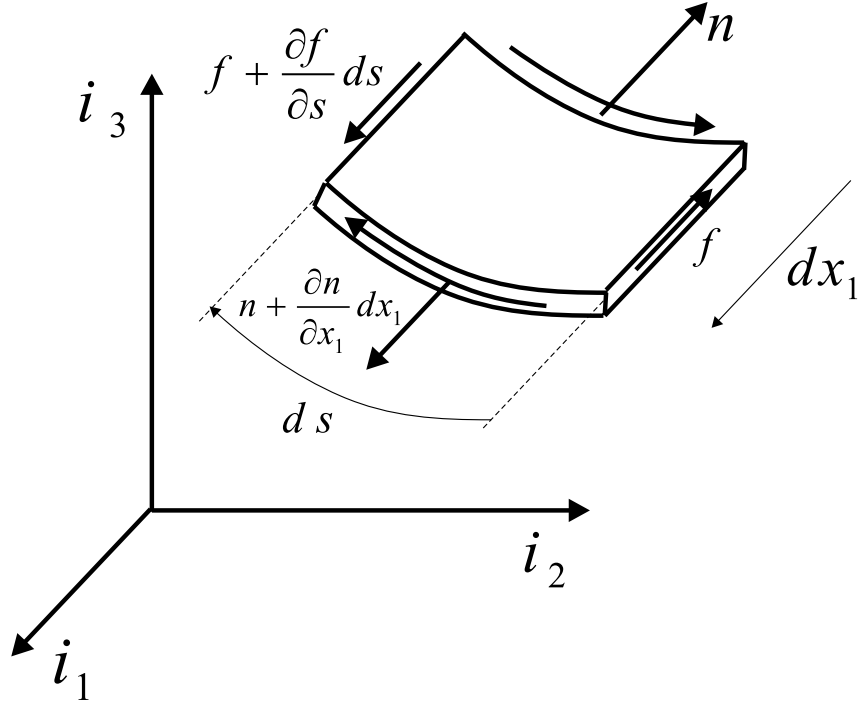


Figure 6.7: Equilibrium of a differential element of the wall.

The height h and width b of curve $\mathcal{C}$ drawn along the mid thickness of the wall is now

$$h = \frac{h_o + h_i}{2}; \quad b = \frac{b_o + b_i}{2}. \quad (6.18)$$

These dimensions are the average height and width of the section. The bending stiffness of the section with respect to the axis $\mathbf{i}_2$ is

$$I_{22} = E \left[\frac{b_o h_o^3}{12} - \frac{b_i h_i^3}{12} \right], \quad (6.19)$$

where E is the material Young's modulus. This expression can be rewritten as

$$I_{22} = \frac{E}{12} [(b + t_w)(h + t_f)^3 - (b - t_w)(h - t_f)^3], \quad (6.20)$$

where the inner and outer dimensions are expressed in terms of the average dimensions and wall thicknesses. Expanding the cubic power and regrouping terms then yields

$$I_{22} = \frac{E}{12} \left\{ 6bh^2t_f \left[1 + \frac{1}{3} \left(\frac{t_f}{h} \right)^2 \right] + 2h^3t_w \left[1 + 3 \left(\frac{t_f}{h} \right) \right] \right\}. \quad (6.21)$$

In view of the thin wall assumption (6.1), the term t_f/h is negligible compared to unity, and the bending stiffness reduces to

$$I_{22} = E \left[2 \frac{t_w h^3}{12} + 2bt_f \left(\frac{h}{2} \right)^2 \right]. \quad (6.22)$$

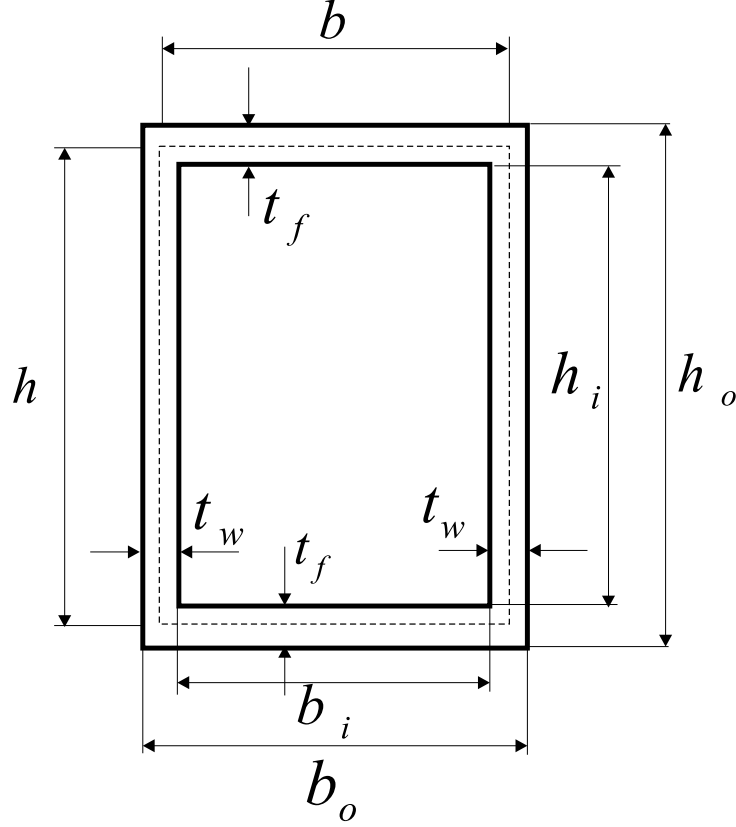


Figure 6.8: Thin-walled rectangular section.

The first term represents the bending stiffness of the two webs, computed with the average height h , whereas the last term gives the contribution of the two flanges using their average width b . The contribution of the flange is given as its bending stiffness about its own centroid $bt_f^3/12$, plus the transport term $bt_f(h/2)^2$. However, the bending stiffness about its own centroid is clearly negligible, leaving the transport term as the sole contribution. A similar reasoning can be used to evaluate the bending stiffness I_{33} .

Fig. 6.9 shows a thin rectangular strip of thickness t and height h . The centroid of this strip is located at distances d_2 and d_3 from axes $\mathbf{i}_3$ and $\mathbf{i}_2$, respectively. The bending stiffnesses of this strip are approximated as

$$I_{22} = E \left[\frac{th^3}{12} + htd_3^2 \right]; \quad I_{33} = E [htd_2^2]; \quad I_{23} = E [htd_2d_3]. \quad (6.23)$$

These results are obtained by first computing the bending stiffness in the principal centroidal axes of the thin strip, then using the transport theorem to translate the bending stiffnesses to the required axis. Terms containing higher powers of the thickness are neglected.

Similar results can be obtained for the same rectangular strip rotated of an angle α with

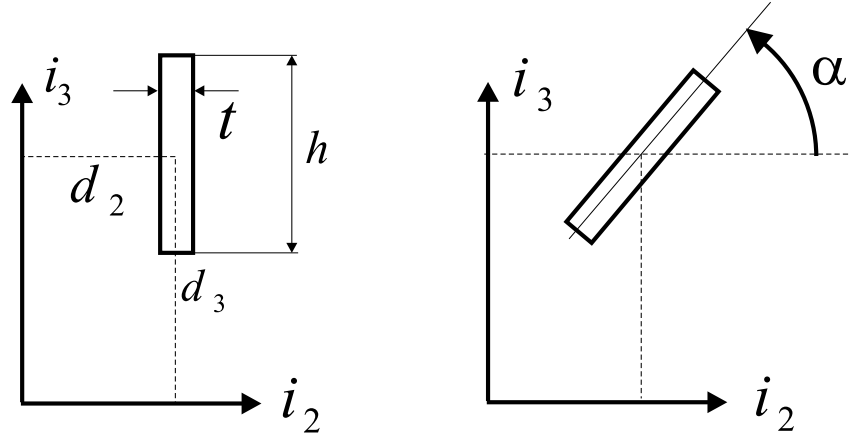


Figure 6.9: A thin rectangular strip.

respect to the axis $\mathbf{i}_2$ also shown in fig. 6.9

$$\begin{aligned} I_{22} &= E \left[\frac{th^3}{12} \sin^2 \alpha + htd_3^2 \right]; & I_{33} &= E \left[\frac{th^3}{12} \cos^2 \alpha + htd_2^2 \right]; \\ I_{23} &= E \left[\frac{th^3}{24} \sin 2\alpha + htd_2d_3 \right]. \end{aligned} \quad (6.24)$$

Example: the bending stiffness of a trapezoidal section

Consider the trapezoidal section shown in fig. 6.15. The bending stiffness about axis $\mathbf{i}_2$ is

$$I_{22} = E \left[\frac{t(2h_1)^3}{12} + \frac{t(2h_2)^3}{12} + 2\frac{tl^3}{12} \sin^2 \alpha + 2tl \left(\frac{h_1 + h_2}{2} \right)^2 \right]; \quad (6.25)$$

where $l = [b^2 + (h_2 - h_1)^2]^{1/2}$ is the length of the upper and lower flanges. The first two terms represent the contribution of the two webs evaluated with the help of (6.23), whereas the last two terms give the contribution of the flanges obtained from (6.24). It is clear that $\sin \alpha = (h_2 - h_1)/l$, and after simplification the bending stiffness becomes

$$I_{22} = \frac{2Et}{3} [h_1^3 + h_2^3 + l(h_1^2 + h_2^2 + h_1h_2)]. \quad (6.26)$$

6.2 Bending of thin-walled beams

Consider a thin-walled beam subjected to axial forces and bending moments, as shown in fig. 6.10. To simplify the formulation, the origin of the axes will be located at the centroid of the section, and the axes $\mathbf{i}_2$ and $\mathbf{i}_3$ are principal axes of bending. The superscripts $(.)^{*c}$ used in section (4.5) will be dropped throughout this chapter to simplify the writing.

The Euler-Bernoulli assumptions discussed in section (3.1) are applicable to thin-walled beams and the distribution of axial stresses, when using the principal centroidal axes of

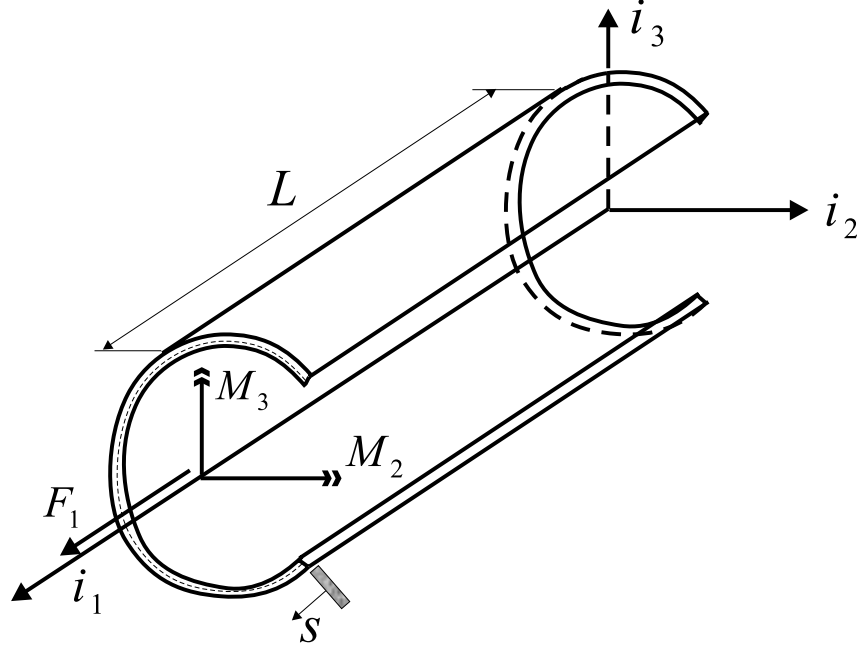


Figure 6.10: Thin-walled beam subjected to axial forces and bending moments.

bending, was found to be (4.30). Hence

$$\sigma_1 = E \left[\frac{F_1}{S} + x_3 \frac{M_2}{I_{22}} - x_2 \frac{M_3}{I_{33}} \right]. \quad (6.27)$$

The axial flow distribution over the cross-section is now

$$n(x_1, s) = E(s)t(s) \left[\frac{F_1(x_1)}{S} + x_3(s) \frac{M_2(x_1)}{I_{22}} - x_2(s) \frac{M_3(x_1)}{I_{33}} \right]. \quad (6.28)$$

This analysis equally applies to open or closed sections as this difference has no impact on the present development.

6.3 Shearing of thin-walled beams

In most practical cases, the bending moments considered in the previous section will act together with shearing forces. These shearing forces give rise to shearing flows that vary over the cross-section. The distribution of this shear flow is evaluated by introducing the axial flow (6.28) into the local equilibrium equation (6.16) to find

$$\frac{\partial f}{\partial s} = -Et \left[\frac{1}{S} \frac{dF_1}{dx_1} + \frac{x_3}{I_{22}} \frac{dM_2}{dx_1} - \frac{x_2}{I_{33}} \frac{dM_3}{dx_1} \right]. \quad (6.29)$$

The global equilibrium eqs. (4.11), (4.13), and (4.15) are introduced to simplify this expression that becomes

$$\frac{\partial f(x_1, s)}{\partial s} = -E(s)t(s) \left[x_2(s) \frac{F_2(x_1)}{I_{33}} + x_3(s) \frac{F_3(x_1)}{I_{22}} \right], \quad (6.30)$$

where the distributed loads p_1 , and moments q_2 and q_3 were assumed to be zero. This first order differential equation can be integrated to evaluate the shear flow distribution corresponding to applied shear forces F_2 and F_3 . However, this integration process differs for open and closed sections.

6.3.1 Shearing of open sections

For an open section, the reciprocity of shearing stresses, eq. (1.9) implies the vanishing of shear flow at the edges of contour $\mathcal{C}$. For instance, the shear flow must vanish at points A and D of the C-channel depicted in fig. 6.11.

In summary, the determination of the shearing flow distribution in a thin-walled beam with an open section under transverse shear forces involves the following procedure.

1. Compute the location of the centroid and the orientation of the principal axes of bending of the section. Select the axes accordingly;
2. Integrate equation (6.30) subjected to the boundary condition $f = 0$ at the edges of the contour.

Example: Shear flow distribution in a C-channel

To illustrate this process, the distribution of shear flow in the C-channel shown in fig. 6.11 will be evaluated. The section has a uniform thickness t , a web depth h , a flange width b , and is subjected to a shearing force F_3 . The origin of the axes is selected at the centroid which is located at a distance $d = b/(2 + h/b)$ from the web midspan. The bending is readily evaluated from eq. (6.23)

$$I_{22} = E \left[\frac{th^3}{12} + 2bt\left(\frac{h}{2}\right)^2 \right] = Et \left(\frac{h^3}{12} + \frac{bh^2}{2} \right) = EtI. \quad (6.31)$$

To ease the algebra, the cross-section is broken into three components, the lower flange, the web, and the upper flange, numbered 1, 2, and 3, respectively. For each part, the origin of the curvilinear variable s is redefined as indicated on the figure. Governing eq. (6.30) becomes

$$\frac{\partial f}{\partial s} = -Et x_3 \frac{F_3}{I_{22}} = -\frac{F_3}{I} x_3. \quad (6.32)$$

For the lower flange this writes

$$\frac{\partial f}{\partial s} = -\frac{F_3}{I} \left(-\frac{h}{2}\right) = \frac{F_3 h}{2I}, \quad (6.33)$$

and integrates to

$$f = \frac{F_3 h}{2I} s + C_1, \quad (6.34)$$

where C_1 is an integration constant. Imposing the boundary condition $f = 0$ at $s = 0$ yields $C_1 = 0$, and the shear flow is found to be linearly distributed over the lower flange

$$f_1 = \frac{F_3 h b}{2I} \left(\frac{s}{b}\right). \quad (6.35)$$

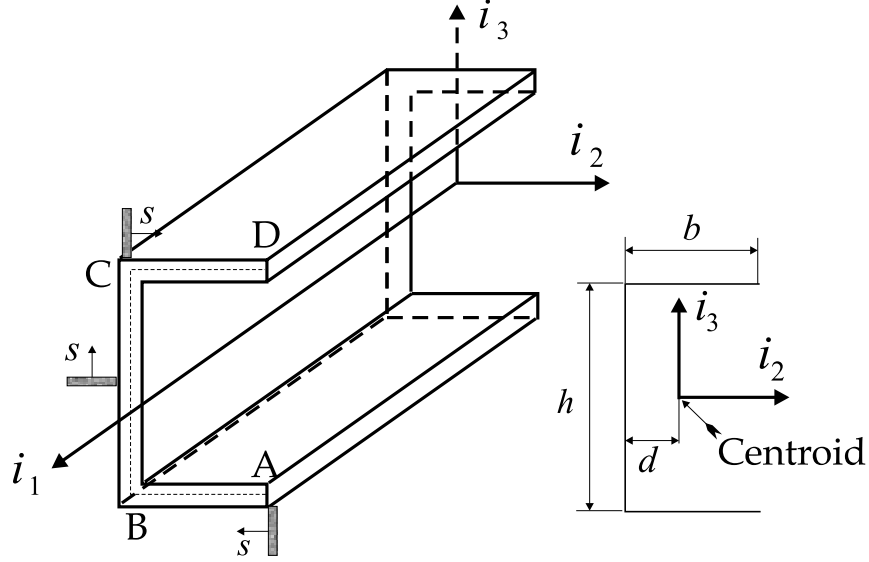


Figure 6.11: Cantilevered beam with a C-channel cross-section.

The subscript $(.)_1$ indicates that this expression describes the shear flow in the lower flange.

For the web, equation (6.32) applies again, but here $x_3 = s$. The shear flow distribution is found by integration and enforcement of the continuity of shear flow at point B

$$f_2 = \frac{F_3 h b}{2I} + \frac{F_3 h^2}{8I} \left[1 - \left(\frac{2s}{h} \right)^2 \right]. \quad (6.36)$$

Finally, the shear flow distribution over the upper flange is found in a similar manner

$$f_3 = \frac{F_3 h b}{2I} \left(1 - \frac{s}{b} \right). \quad (6.37)$$

This distribution satisfies the continuity condition at point C and the boundary condition $f_3 = 0$ at $s = b$. The overall distribution of shear flow is depicted in fig. 6.12. The maximum shear flow occurs at the middle of the web and is

$$f_{max} = \frac{F_3 h^2}{8I} \left(1 + \frac{4b}{h} \right). \quad (6.38)$$

The resultant shear forces acting on the various parts of the section can be evaluated over the lower flange, web, and upper flange, respectively

$$\int_0^b f_1 ds = \frac{F_3 h b^2}{4I} = R; \quad \int_{-h/2}^{h/2} f_2 ds = F_3; \quad \int_0^b f_3 ds = \frac{F_3 h b^2}{4I} = R. \quad (6.39)$$

It is now possible to check the overall equilibrium of the section. The forces along the $\mathbf{i}_2$ direction sum up to zero, as it should be since no load was applied in that direction. The force F_3 in the web corresponds to the applied shear force. Finally, there clearly exists a torque

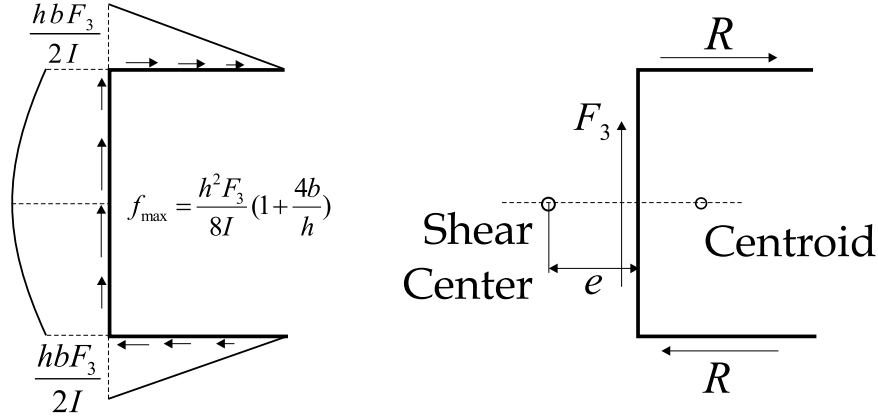


Figure 6.12: Distribution of shear flow over the C-channel cross-section.

resultant about the origin of the axes. However, the sole shear force F_3 was applied, though its line of action was not specified. For overall equilibrium of the section to be satisfied, the shearing force F_3 must be applied at a point K , such that the net torque vanishes about that point. This condition is

$$M_{1K} = eF_3 - Rh = 0. \quad (6.40)$$

The distance e from the web to point K is found as

$$e = \frac{Rh}{F_3} = \frac{h^2 b^2}{4I} = \frac{3b/h}{1 + 6b/h} b. \quad (6.41)$$

Point K is called the *shear center* of the cross-section. If the transverse shearing force is applied at the shear center, the overall equilibrium of the section is satisfied, and the beam is subjected to bending and shearing only, i.e. the beam does not twist.

6.3.2 Shearing of closed sections

In the case of closed sections, eq. (6.30) still applies though no boundary condition is readily available. One notable exception occurs when the section presents an axis of symmetry, as shown in fig. 6.13. The shear force F_2 acts in the $\mathbf{i}_1 \mathbf{i}_2$ plane which is a plane of symmetry of the section. This symmetry implies the vanishing of the shear flow at points A and B . Hence, the upper and lower halves of the section can be analyzed separately, as if they were two independent open sections. However, if a shear force F_3 is applied, the above symmetry argument is no longer applicable, and no boundary condition is available to integrate eq. (6.30). To overcome this problem, the following reasoning is devised.

The first step consists of cutting the beam along its axis at an arbitrary point of the section, as shown in fig. 6.14. The shear flow distribution $f_0(s)$ for this auxiliary problem is readily found using the procedure described in the previous section. This shear flow deforms each differential element of the section, generating a shearing angle γ_s that in turns, causes

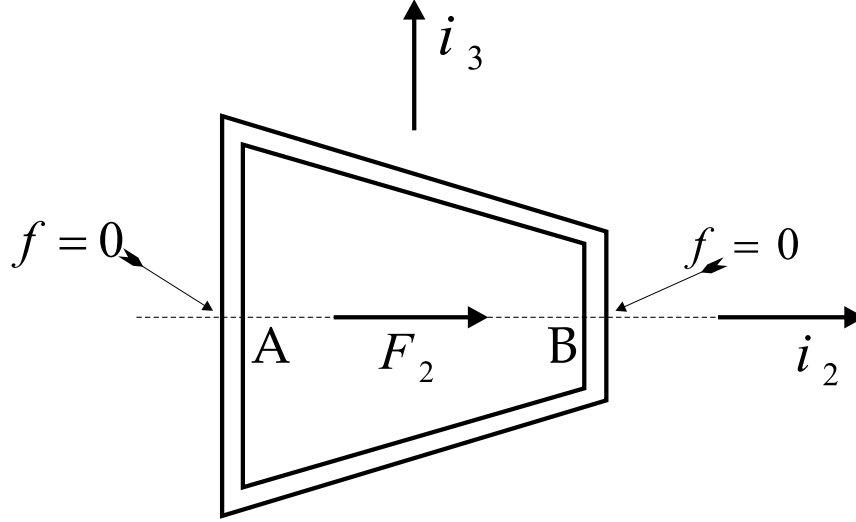


Figure 6.13: Trapezoidal section subjected to a shear force.

an axial displacement du_1 . An inspection of fig. 6.14 yields

$$du_1 = \gamma_s ds = \frac{\tau_s}{G} ds = \frac{f_0(s)}{Gt} ds. \quad (6.42)$$

The total relative axial displacement at the cut u_0 is then

$$u_0 = \int_C \frac{f_0(s)}{Gt} ds. \quad (6.43)$$

This relative displacement would appear in the auxiliary problem. However, for the closed section problem no such relative displacement exists. Therefore, a *closing shear flow* f_c is applied at the cut so as to zero this relative displacement. At all points of the section the constant value of this closing shear flow is added to the existing shear flow f_0 to yield at total shear flow $f_t(s) = f_0(s) + f_c$. The corresponding total relative axial displacement at the cut is then

$$u_t = \int_C \frac{f_0(s) + f_c}{Gt} ds. \quad (6.44)$$

The closing shear flow is evaluated by requiring the vanishing of the relative displacement. Imposing $u_t = 0$ yields

$$f_c = - \frac{\int_C \frac{f_0(s)}{Gt} ds}{\int_C \frac{1}{Gt} ds}. \quad (6.45)$$

The procedure used to compute the shear flow distribution in a closed section is summarized by the following steps.

1. Compute the shear flow distribution $f_0(s)$ for an auxiliary problem obtained by cutting the beam along its axis at an arbitrary point of the section. The procedure described in section (6.3.1) is used;

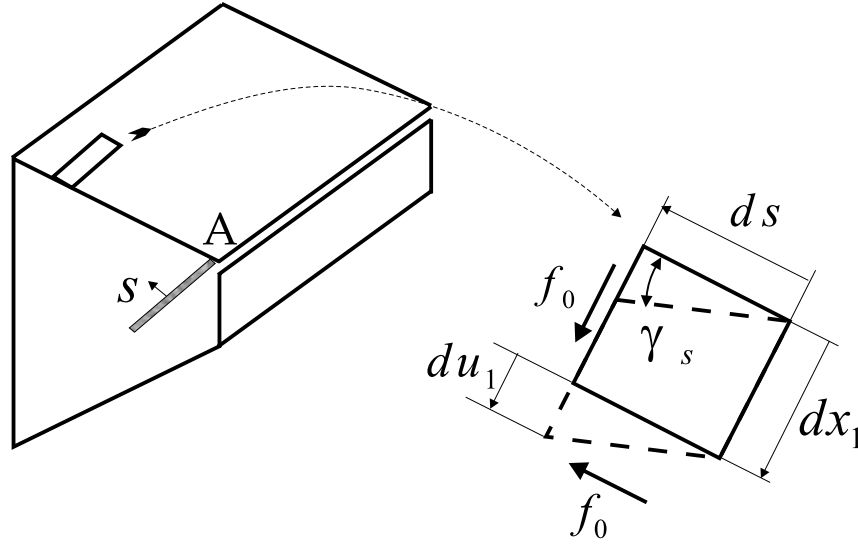


Figure 6.14: Trapezoidal section with a cut at point A.

2. Compute the closing shear flow f_c from eq. (6.45);
3. The shear flow distribution in the closed section is $f_t(s) = f_0(s) + f_c$.

Example: Shear flow distribution in a trapezoidal section

Consider, as an example, the trapezoidal section shown in fig. 6.15. The wall thickness t is constant and the section has been broken into four components, the upper flange, left web, lower flange, and right web, numbered 1,2 ,3 and 4, respectively, each with its own origin for the curvilinear variable s . The bending stiffness of the section is given by eq. (6.26), and writes

$$I_{22} = EtI; \quad I = \frac{2}{3} [h_1^3 + h_2^3 + (h_1^2 + h_2^2 + h_1 h_2)l].$$

The first step of the procedure is to cut the beam at an arbitrary point, say A . The distribution of shear flow in the open section is found by integrating differential eq. (6.30). For the upper flange, the boundary condition is $f = 0$ at $s = 0$. The shear flow in the other components of the section is obtained by integration of the same equation and enforcement of continuity of shear flow at points B , C , and D . The complete shear flow distribution is

$$\begin{aligned} f_{01} &= \frac{F_3}{I} \left[\frac{h_2 - h_1}{2l} s^2 - h_2 s \right]; & f_{02} &= \frac{F_3}{2I} [s^2 - h_1^2 - (h_1 + h_2)l]; \\ f_{03} &= \frac{F_3}{I} \left[\frac{h_2 - h_1}{2l} s^2 + h_1 s - \frac{h_1 + h_2}{2} l \right]; & f_{04} &= \frac{F_3}{2I} [-s^2 + h_2^2]. \end{aligned} \quad (6.46)$$

The next step of the procedure requires the evaluation of the closing shear flow according to equation (6.45). The following integral are computed

$$\int_c \frac{f_0}{Gt} ds = \int_0^l \frac{f_{01}}{Gt} ds + \int_{-h_1}^{+h_1} \frac{f_{02}}{Gt} ds + \int_0^l \frac{f_{03}}{Gt} ds + \int_{-h_2}^{+h_2} \frac{f_{04}}{Gt} ds,$$

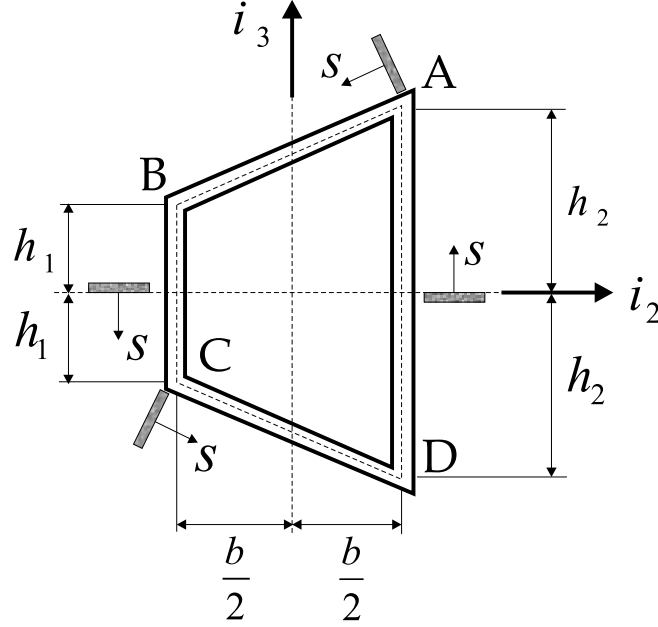


Figure 6.15: Thin-walled trapezoidal section.

or

$$\int_C \frac{f_0}{Gt} ds = -\frac{F_3}{3GtI} [2(h_1^3 - h_2^3) + (h_1 + 2h_2)l^2 + 3(h_1 + h_2)lh_1];$$

and

$$\int_C \frac{1}{Gt} ds = \int_0^l \frac{ds}{Gt} + \int_{-h_1}^{+h_1} \frac{ds}{Gt} + \int_0^l \frac{ds}{Gt} + \int_{-h_2}^{+h_2} \frac{ds}{Gt} = \frac{2(l + h_1 + h_2)}{Gt},$$

to yield the closing shear flow

$$f_c = \frac{F_3}{I} \frac{2(h_1^3 - h_2^3) + (h_1 + 2h_2)l^2 + 3(h_1 + h_2)lh_1}{6(l + h_1 + h_2)}. \quad (6.47)$$

The final distribution of shear flow is found by adding this closing shear flow to the shearing flow distribution for the open section (6.46). This distribution for the closed section is depicted in fig. 6.16 which shows that the maximum shear flow is found at the middle of the right web. Also shown on this figure are the shear force resultant on each component of the section. Clearly, the horizontal forces sum up to zero since no shear force was applied in that direction. The summation of the forces in the vertical direction gives

$$\begin{aligned} \frac{F_3}{I} \left\{ \left[\left(\frac{h_1 + 2h_2}{6} l^2 - f_c l \right) \frac{h_2 - h_1}{l} \right] + \left[\frac{2h_1^3}{3} + (h_1 + h_2)lh_1 - 2f_c h_1 \right] \right. \\ \left. + \left[\left(\frac{h_1 + 2h_2}{6} l^2 - f_c l \right) \frac{h_2 - h_1}{l} \right] + \left[\frac{2h_2^3}{3} + 2f_c h_2 \right] \right\} = \frac{F_3}{I} I = F_3. \end{aligned}$$

The distributed shear flow exactly sums up to the applied shear force F_3 .

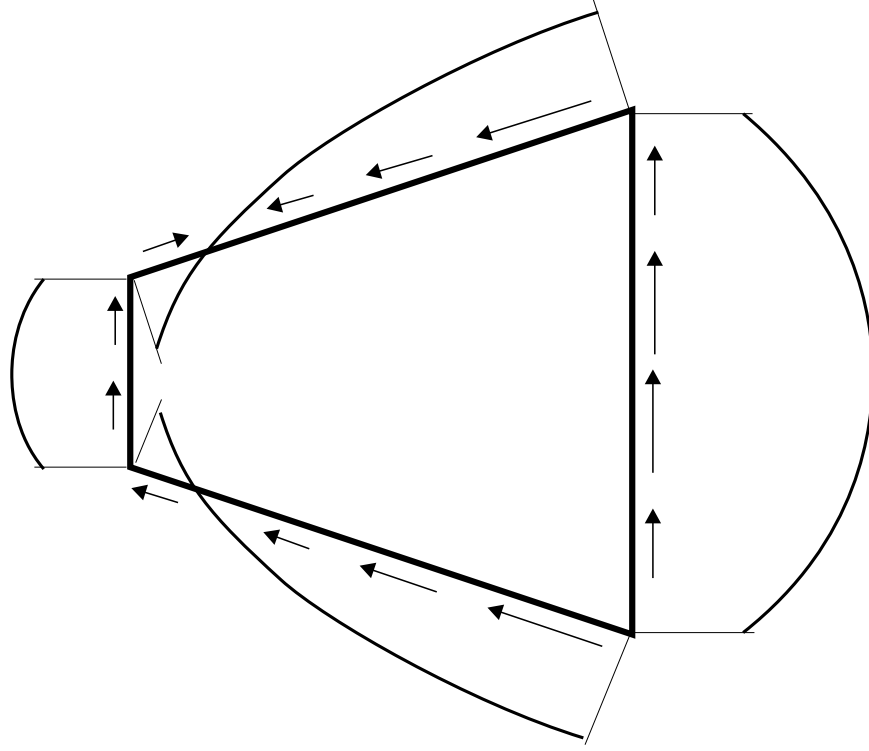


Figure 6.16: Shear flow distribution in the trapezoidal section.

6.3.3 Shearing of multi-cellular sections

Multi-cellular sections are common in aeronautical constructions. Fig. 6.17 shows a typical wing section with two cells. Here again, the shearing flow distribution satisfies equation (6.30), but no boundary condition is available. To remedy this situation, a procedure similar to that used for the closed section will be used.

The multi-cellular beam is cut along its axis at arbitrary points. One cut per cell is required to eliminate all the closed paths of the section. Fig. 6.17 shows the two cuts corresponding to the two cells present in that example. The shear flow distribution in the resulting open section is evaluated with the procedure described in section (6.3.1). Let $f_{01}(s)$, $f_{02}(s)$, and $f_{03}(s)$ be the shear flow distributions along C_1 , C_2 , and C_3 , respectively. The next step is to apply a closing shear flow to zero the relative axial displacement at each cut. These conditions are

$$\begin{aligned} u_{1t} &= \int_{C_1} \frac{f_{01}(s) + f_{c1}}{Gt} ds + \int_{C_3} \frac{f_{03}(s) + (f_{c1} + f_{c2})}{Gt} ds = 0; \\ u_{2t} &= \int_{C_2} \frac{f_{02}(s) + f_{c2}}{Gt} ds + \int_{C_3} \frac{f_{03}(s) + (f_{c1} + f_{c2})}{Gt} ds = 0, \end{aligned} \quad (6.48)$$

for the left and right cells, respectively. These equations can be recast as a set of two linear

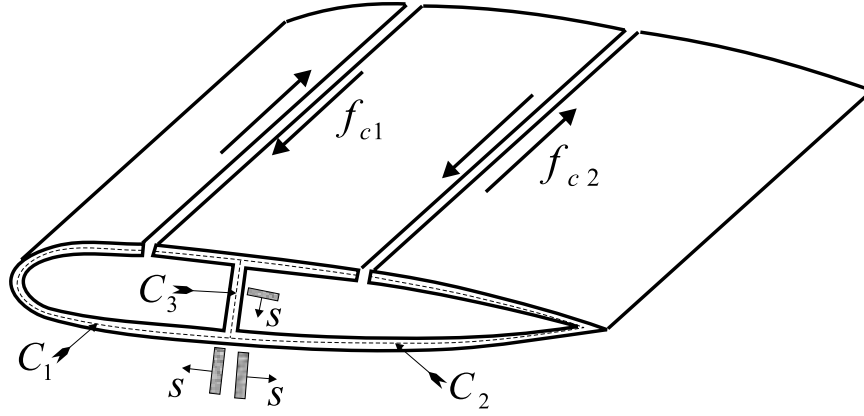


Figure 6.17: A thin-walled, multi-cellular section.

equations for the unknown closing shear flows f_{c1} and f_{c2}

$$\begin{aligned} \left[\int_{C_1+C_3} \frac{1}{Gt} ds \right] f_{1c} + \left[\int_{C_3} \frac{1}{Gt} ds \right] f_{2c} &= - \int_{C_1+C_3} \frac{f_0(s)}{Gt} ds; \\ \left[\int_{C_3} \frac{1}{Gt} ds \right] f_{1c} + \left[\int_{C_2+C_3} \frac{1}{Gt} ds \right] f_{2c} &= - \int_{C_2+C_3} \frac{f_0(s)}{Gt} ds. \end{aligned} \quad (6.49)$$

The shear flow in the multi-cellular section is then found by adding the closing shear flows to the shear flows in the open section, i.e. $f_{01}(s) + f_{c1}$, $f_{02}(s) + f_{c2}$, and $f_{03}(s) + (f_{c1} + f_{c2})$ for curves C_1 , C_2 , and C_3 , respectively.

The procedure is readily extended to multi-cellular section possessing N cells. First, the multi-cellular section is transformed into an open section by creating N cuts, one per cell. The shear flow distribution in the resulting open section is then evaluated with the help of the procedure of section (6.3.1). Next, unknown closing shear flows are applied at each cut so as to zero the relative axial displacements at these locations. These conditions yield a set of N simultaneous equations for the N closing shear flows. Finally, the shear flow distribution in the multi-cellular section is found by adding the closing shear flows to the shear flows for the open section.

6.3.4 Problems

Problem 6.1

The cross-section of a high lift device is shown in fig 6.18. The aerodynamic pressure acting on the lower panel of the device has a net resultant $F_3 = 100 \times 10^3 \text{ N}$ and its line of action is aligned with axis $\mathbf{i}_3$, as indicated on the figure. Material properties are: $E = 73 \text{ GPa}$, $G = 30 \text{ GPa}$.

1. Find the distribution of shear stress flow associated with the shear force F_3 .
2. Sketch this shear flow distribution.
3. determine the location and magnitude of the maximum shear stress.

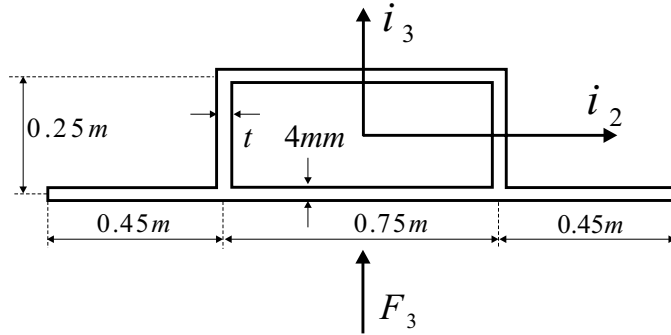


Figure 6.18: High lift device subjected to a transverse shear force.

6.4 The shear center

The *shear center* of a section is defined as the point at which transverse shearing forces must be applied for the beam to bend without twisting. The determination of the location of the shear center is a crucial step in the analysis of thin-walled beams subjected to transverse shearing forces. Indeed, the procedure for computing the shear flow distribution associated with an applied shear force given in the previous section assumes the application of the sole shear force, i.e. no torque is applied. If the shear force is not applied at the shear center, it will generate torsion in the beam, and the shear flow associated with this torsion must be added to that associated with shearing.

The investigation of the C-channel described in section (6.3.1) showed that the shear flow distribution associated with an applied shear force does present a resulting torque about all points of the section except the shear center. The same argument will be used to determine the location of the shear center of general, thin-walled sections. It will be found as the point about which the resultant torque associated with the shear flow distribution generated by an applied shear force vanishes.

Let $\mathcal{G}_2(s)$ and $\mathcal{G}_3(s)$ be the shear flow distributions corresponding to unit shear forces $F_{2(2)} = 1$, $F_{3(2)} = 0$, and $F_{2(3)} = 0$, $F_{3(3)} = 1$, respectively. For (6.30) these distributions satisfy the following differential equations

$$\frac{d\mathcal{G}_2}{ds} = -\frac{Et}{I_{33}} x_2; \quad \frac{d\mathcal{G}_3}{ds} = -\frac{Et}{I_{22}} x_3. \quad (6.50)$$

Of course, the procedure for the solution of the equations depends on whether the section is open, closed, or multi-cellular.

It is interesting to compute the shear force resultant associated with $\mathcal{G}_2$ and $\mathcal{G}_3$. For instance, the shear force $F_{2(2)}$ associated with $\mathcal{G}_2$ is found from eq. (6.10) as

$$F_{2(2)} = \int_c \mathcal{G}_2 \frac{dx_2}{ds} ds. \quad (6.51)$$

Integrating by parts then yields

$$F_{2(2)} = - \int_c x_2 \frac{d\mathcal{G}_2}{ds} ds + [x_2 \mathcal{G}_2]_{\text{boundary}}. \quad (6.52)$$

The boundary term always vanishes: if the section is open, $\mathcal{G}_2 = 0$ at the edges of the section, and when the section is closed or multi-cellular no boundaries exist. Introducing the governing eq. (6.50) for the shear flow distribution then leads to

$$F_{2(2)} = \int_C \frac{Et}{I_{33}} x_2^2 ds = \frac{1}{I_{33}} \int_C E x_2^2 t ds = \frac{I_{33}}{I_{33}} = 1. \quad (6.53)$$

The shear force $F_{3(2)}$ associated with $\mathcal{G}_2$ is readily evaluated using a similar procedure. From (6.10) the shear force F_3 is

$$F_{3(2)} = \int_C \mathcal{G}_2 \frac{dx_3}{ds} ds = - \int_C x_3 \frac{\mathcal{G}_2}{ds} ds + [x_3 \mathcal{G}_2]_{\text{boundary}}, \quad (6.54)$$

where the boundary term vanishes for the reason given earlier. Introducing the governing equation (6.50) then yields

$$F_{3(2)} = \int_C \frac{Et}{I_{33}} x_2 x_3 ds = \frac{1}{I_{33}} \int_C E x_2 x_3 t ds = \frac{I_{23}}{I_{33}} = 0, \quad (6.55)$$

where the cross bending stiffness $I_{23} = 0$ because the axes were selected as the principal axes of bending. Results (6.53) and (6.55) were expected: the shear force resultant associated with $\mathcal{G}_2$ are $F_{2(2)} = 1$ and $F_{3(2)} = 0$, since $\mathcal{G}_2$ was specifically computed for that applied loading. Similarly, it can be shown that the shear force resultant associated with $\mathcal{G}_3$ are $F_{2(3)} = 0$ and $F_{3(3)} = 1$, as expected.

Next, the torque computed about the origin of the axes associated with $\mathcal{G}_2$ is evaluated with the help of eq. (6.11)

$$M_{1O} = \int_C \mathcal{G}_2 r_o ds. \quad (6.56)$$

In general, this torque is not zero. However, by definition, the torque computed about the shear center should vanish, i.e.

$$M_{1K} = \int_C \mathcal{G}_2 r_k ds = 0. \quad (6.57)$$

Introducing expression (6.14) for r_k yields

$$-x_{2k} \int_C \mathcal{G}_2 \frac{dx_3}{ds} ds + x_{3k} \int_C \mathcal{G}_2 \frac{dx_2}{ds} ds = - \int_C \mathcal{G}_2 r_o ds. \quad (6.58)$$

The first two integral correspond to the shear force resultants $F_{2(2)}$ and $F_{3(2)}$ associated with the shear flow distribution $\mathcal{G}_2$. These resultant were evaluated in eqs. (6.53) and (6.55), respectively. Hence,

$$x_{3k} = - \int_C \mathcal{G}_2 r_o ds. \quad (6.59)$$

A similar reasoning for the shear flow distribution $\mathcal{G}_3$ yields the other coordinate of the shear center

$$x_{2k} = \int_C \mathcal{G}_3 r_o ds. \quad (6.60)$$

The procedure for the determination of the location of the shear center is summarized in the following steps.

1. Compute the location of the centroid and the orientation of the principal axes of bending. Select the axes accordingly;
2. Compute the shear flow distributions $\mathcal{G}_2$ corresponding to a unit shear force $F_{2(2)} = 1$, $F_{3(2)} = 0$.
3. Compute the shear flow distributions $\mathcal{G}_3$ corresponding to a unit shear force $F_{2(3)} = 0$, $F_{3(3)} = 1$. Of course, the procedure used to determine these shear flows depend on the geometric nature of the cross-section, as discussed in section (6.3);
4. Compute the location of the shear center using eqs. (6.59) and (6.60).

Example: The shear center of a trapezoidal section

This procedure is illustrated by computing the location of the shear center for the closed trapezoidal section described in section (6.3.2). The distribution of shear flow was found as the sum of the shear flow in the open section (6.46) plus the closing shear flow (6.47) for an applied shear force F_3 . The location of the shear center then follows from (6.60)

$$x_{2k} = \int_C (\bar{f}_0(s) + \bar{f}_c) r_o \, ds, \quad (6.61)$$

where the barred quantities are found by setting $F_3 = 1$ in the corresponding expressions (6.46) and (6.47). Evaluation of the integral yields

$$x_{2k} = \frac{b}{4} \frac{h_2 - h_1}{l} \frac{1 - \frac{h_1 + h_2}{l}}{1 + \frac{h_1 + h_2}{l}} \frac{1 + l \frac{h_2^2 - h_1^2}{h_2^3 - h_1^3}}{1 + \frac{h_2 - h_1}{l} \frac{h_2^3 + h_1^3}{h_2^3 - h_1^3}}. \quad (6.62)$$

Due to the symmetry of the problem, the other coordinate of the shear center is $x_{3k} = 0$. It is clear that when $h_2 = h_1$ the trapezoidal section becomes rectangular, and $x_{2k} = 0$, as required by symmetry.

6.4.1 Problems

Problem 6.2

In section 6.4 a procedure was derived for the determination of the location of the shear center. The derivation was developed by first selecting the axis system to be the principal centroidal axis system. Such a system always exists, and hence, the derivation presented in this section is general. However, it is not always convenient to use the principal centroidal axes. Derive a procedure for the determination of the location of the shear center using an arbitrary axis system. Hint: first derive the governing equations for the shear flow distributions $\mathcal{G}_2$ and $\mathcal{G}_3$ corresponding to unit shear forces along the $\mathbf{i}_2$ and $\mathbf{i}_3$, respectively.

Problem 6.3

Fig. 5.20 depicts the thin-walled, semi-circular open cross-section of a beam. The wall thickness is t , and the material Young's and shearing moduli are E and G , respectively. Find the location of the shear center of the section. Note: It is more convenient to work with the angle θ as a variable describing the geometry of the section: $s = R\theta$, $ds = R d\theta$.

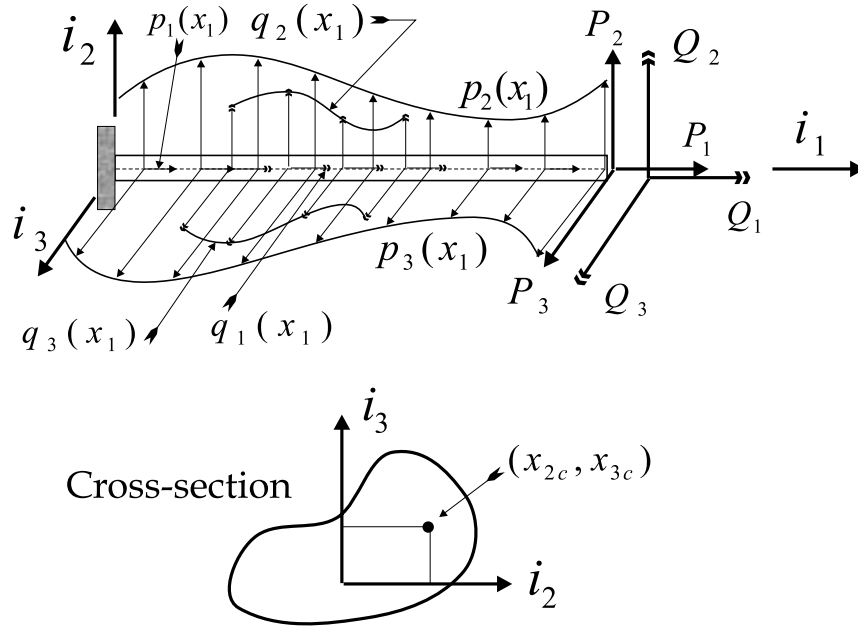


Figure 6.19: Beam under a complex loading condition.

Problem 6.4

Fig. 5.21 depicts the cross-section of a thin-walled beam. Compute the location of the shear center for this section.

6.5 Coupled bending-torsion problems

In chapter 4, the response of a beam with an arbitrary cross-section subjected to a complex loading condition was analyzed. The loading involved distributed and concentrated axial and transverse loads, as well as distributed and concentrated moments. However, two important restrictions were made: no torques were applied, and the transverse shearing forces were applied in such a way that the beam bends without twisting. The first restriction is easily removed. Indeed, if torques are applied, the beam will twist and the analysis tools developed in chapter 5 readily apply to this problem.

The knowledge of the shear center location allows the removal of the second restriction. Applied transverse forces will bend the beam without twisting if and only if they are applied at the shear center. In other words, if all transverse loads are applied at the shear center, the analysis developed in chapter 4 is applicable. If a transverse load is not applied at the shear center, it can always be replaced by an equal transverse load applied at the shear center plus an equivalent applied torque. This equivalence stems from the assumption of infinite rigidity of the cross-section in its own plane. To be more precise, let the transverse distributed load p_2 be applied at a point A of the section with coordinates (x_{2a}, x_{3a}) . This transverse load is then equivalent to a transverse load p_2 applied at the shear center plus a distributed torque $-(x_{3a} - x_{3c})p_2$.

Consider now the response of the beam with an arbitrary cross-section subjected to the general loading depicted in fig. 6.19. Note the presence of distributed and concentrated torques, q_1 and Q_1 , respectively. The above remarks lead to the following procedure.

1. Compute the centroid of the section;
2. Compute the orientation of the principal axes of bending $\mathbf{i}_1^*$, $\mathbf{i}_2^*$, and $\mathbf{i}_3^*$, and the principal centroidal bending stiffnesses;
3. Compute the location of the shear center, see section (6.4);
4. Compute the torsional stiffness, see chapter 5;
5. Solve the extensional problem (4.31), with appropriate boundary conditions;
6. Solve two decoupled bending problems eqs. (4.32) and (4.33), with appropriate boundary conditions;
7. Solve the torsional problem governed by the following differential equation

$$\frac{d}{dx_1^*} \left(J \frac{d\Phi_1^*}{dx_1^*} \right) = - [q_1^* - p_2^*(x_{3a}^* - x_{3k}^*) + p_3^*(x_{2a}^* - x_{2k}^*)]; \quad (6.63)$$

subjected to boundary conditions at the root

$$\Phi_1^* = 0, \quad \Phi_1^* = 0; \quad (6.64)$$

and at the tip

$$\Phi_1^* = L, \quad J \frac{d\Phi_1^*}{dx_1^*} = Q_1^* - P_2^*(x_{3a}^* - x_{3k}^*) + P_3^*(x_{2a}^* - x_{2k}^*); \quad (6.65)$$

The knowledge of the centroid and shear center locations, as well as the orientation of the principal axes of bending allows a complete decoupling of the problem into four independent problems: an axial problem, two bending problems, and a torsional problem.

Example: Wing structure subjected to aerodynamic lift and moment

An important example of this procedure is the wing bending torsion coupled problem shown in fig. 6.20. The principal axes of bending $\mathbf{i}_2$ and $\mathbf{i}_3$ are selected with their origin at the shear center. Axis $\mathbf{i}_1$ is along the locus of the shear centers of all the cross-sections assumed to form a straight line called the *elastic axis*. The aerodynamic loading consists of a lift per unit span L_{AC} applied at the aerodynamic center, and an aerodynamic moment per unit span M_{AC} . Note that for our choice of sign convention, this nose-up aerodynamic moment is a negative quantity. The differential equation for bending in plane ($\mathbf{i}_1 \mathbf{i}_3$) is

$$\frac{d^2}{dx_1^2} \left(I_{22} \frac{d^2 U_3}{dx_1^2} \right) = L_{AC},$$

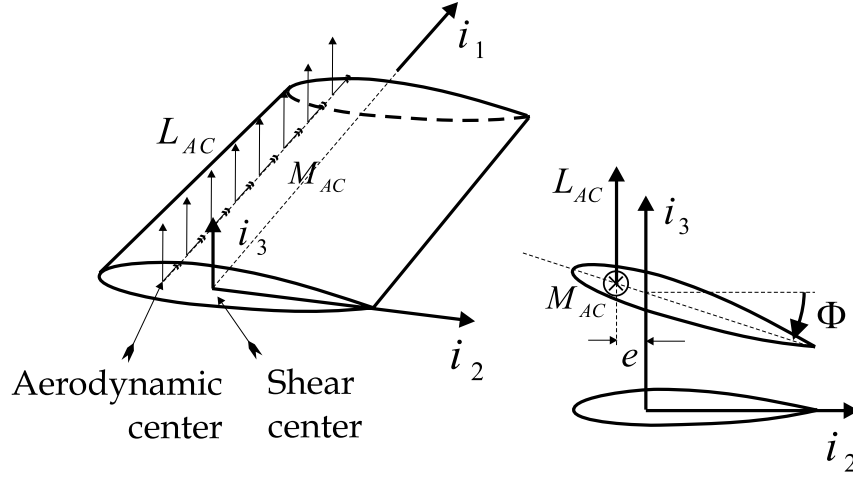


Figure 6.20: The wing bending torsion coupled problem.

subjected to boundary conditions at the clamped root of the wing

$$@x_1 = 0, \quad U_3 = \frac{dU_3}{dx_1} = 0,$$

and at its tip

$$@x_1 = L, \quad \frac{d^2U_3}{dx_1^2} = \frac{d^3U_3}{dx_1^3} = 0.$$

The governing equation of torsion is

$$\frac{d}{dx_1} \left(J \frac{d\Phi_1}{dx_1} \right) = -(-M_{AC} - eL_{AC});$$

subjected to boundary conditions at the root and tip

$$@x_1 = 0, \quad \Phi_1 = 0; \quad @x_1 = L, \quad J \frac{d\Phi_1}{dx_1} = 0,$$

where e is the distance between the aerodynamic center and the shear center. Note that for symmetric airfoils $M_{AC} = 0$, but the wing still twists because the lift is applied at the aerodynamic center which does not coincide with the shear center. For typical transport aircrafts, the aerodynamic and shear centers are located at 25% and 35% chord, respectively. As a result the lift generates a nose-up torque on the wing.

6.6 Torsion of thin-walled beams

The analysis developed in chapter 5 applies to bars with arbitrary cross-sections and is thus very general. Unfortunately, a partial differential equation must be solved to obtain the warping or stress function and the corresponding stress distribution over the cross-section. However, in the case of thin-walled beams, closed form solutions can be obtained.

6.6.1 Torsion of open sections

The torsional behavior of thin-walled beams with open section was investigated in section (5.6) with the help of the membrane analogy. The shear stress is linearly distributed through the thickness of the wall, and the torsional stiffness is proportional to the cube of the wall thickness. Hence, thin-walled open sections have very limited torque carrying capability.

6.6.2 Torsion of closed section

Consider now a thin-walled tube with a closed section of arbitrary shape subjected to an applied torque, as depicted in fig. 6.21. The cross-section consists of a single closed cell defining a contour $\mathcal{C}$. As was the case for the Saint-Venant solution, eq. (5.3.1), the beam is assumed to be in a state of uniform torsion, i.e. the twist rate is constant along the span, and the axial strain and stress components vanish. Hence, the axial flow vanishes, and the local equilibrium eq. (6.16) implies

$$\frac{\partial f}{\partial s} = 0. \quad (6.66)$$

The shear flow must remain constant over the contour $\mathcal{C}$, i.e.

$$f(s) = f = \text{constant}. \quad (6.67)$$

This constant shear flow distribution generates a torque M_1 about the origin of the axes given by (6.11)

$$M_1 = \int_{\mathcal{C}} f(s) r_o(s) ds = f \int_{\mathcal{C}} r_o(s) ds, \quad (6.68)$$

where the last equality follows from the constancy of the shear flow. The integral of the moment arm over the contour equals twice the area $\mathcal{A}$ enclosed by $\mathcal{C}$. Hence,

$$M_1 = 2\mathcal{A}f. \quad (6.69)$$

From the definition of the shear flow, eq. (6.7), the shear stress τ_s resulting from a torque M_1 is

$$\tau_s(s) = \frac{M_1}{2\mathcal{A}t(s)}. \quad (6.70)$$

Under the effect of the applied shear stress, each differential element of the wall deforms as depicted in fig. 6.22, creating a shearing strain $\gamma_s(s)$. The strain energy stored in this differential element is $\frac{1}{2}\gamma_s\tau_s t ds dx_1$. The total strain energy for the entire contour is then

$$E = \frac{1}{2} \int_{\mathcal{C}} \gamma_s \tau_s t ds dx_1 = \frac{1}{2} \int_{\mathcal{C}} \frac{\tau_s^2}{G} t ds dx_1, \quad (6.71)$$

where the material constitutive law, eq. (1.91) was used. Introducing the shear stress distribution (6.70) then yields

$$E = \frac{1}{2} \frac{M_1^2}{4\mathcal{A}^2} \int_{\mathcal{C}} \frac{ds}{Gt} dx_1. \quad (6.72)$$

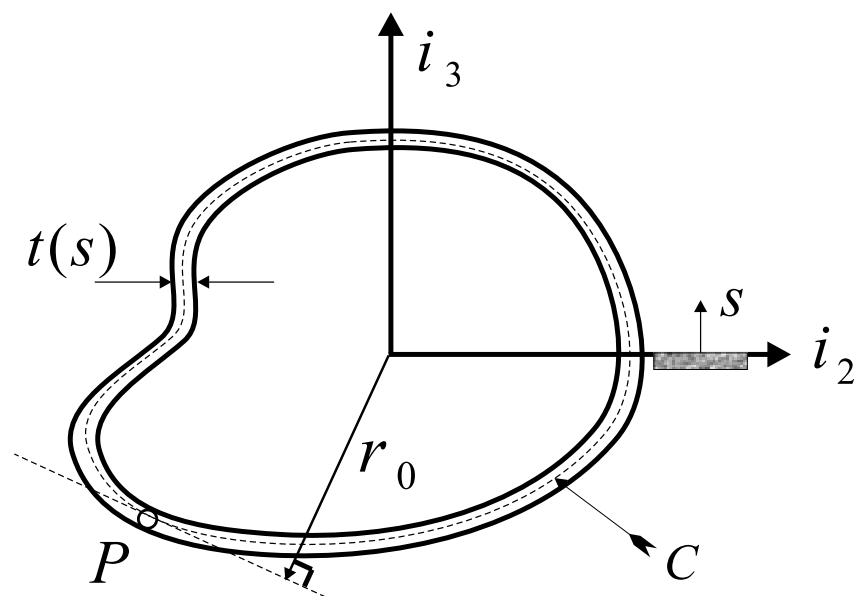


Figure 6.21: Thin walled tubes of arbitrary cross-sectional shape.

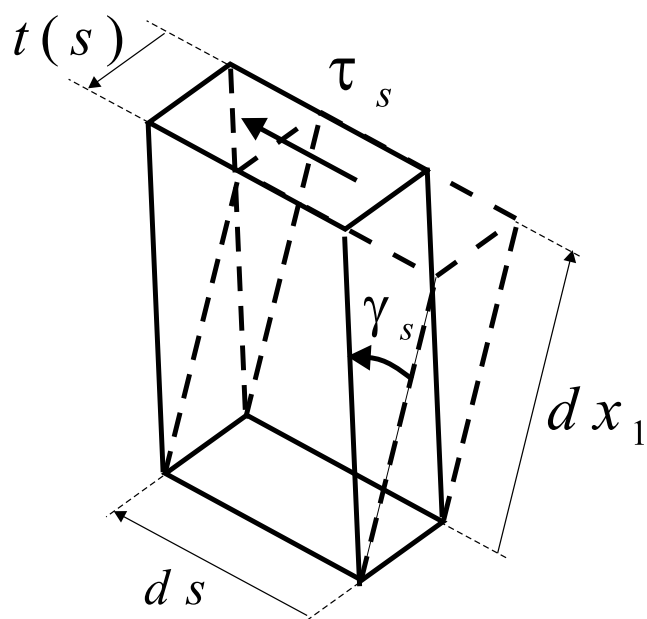


Figure 6.22: Deformation of a differential element of the wall.

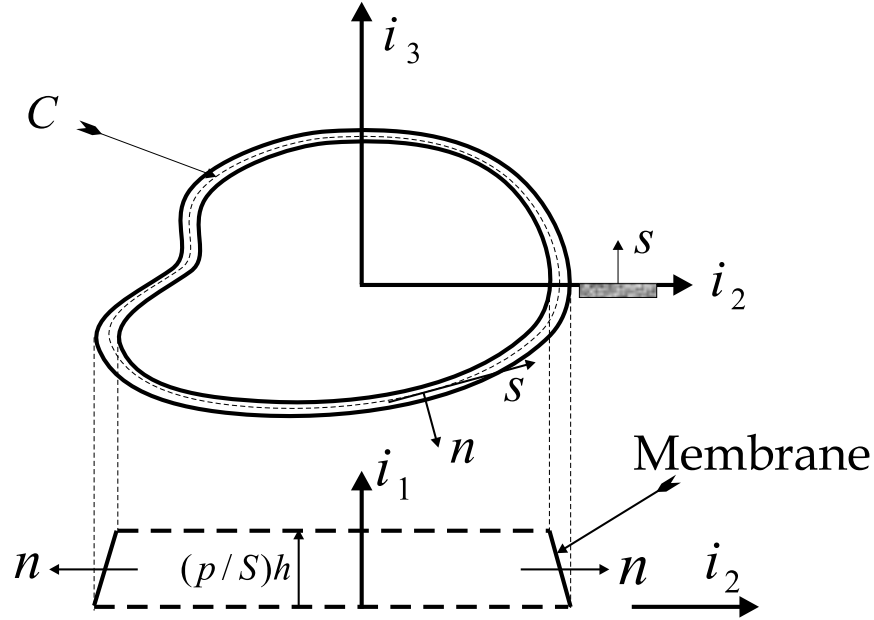


Figure 6.23: A thin membrane stretched over a thin-wall section.

On the other hand, this stored energy must equal the work done by the torque that deforms the section, i.e.

$$E = \frac{1}{2} M_1 \frac{d\Phi_1}{dx_1} dx_1 = \frac{1}{2} M_1 \kappa_1 dx_1. \quad (6.73)$$

We now identify expression (6.72) and (6.73) for the energy to find

$$\kappa_1 = \frac{M_1}{4\mathcal{A}^2} \int_C \frac{ds}{Gt}. \quad (6.74)$$

This relationship expresses a proportionality between the torque and the resulting twist rate. The constant of proportionality is the torsional stiffness which becomes

$$J = \frac{4\mathcal{A}^2}{\int_C \frac{ds}{Gt}}. \quad (6.75)$$

For an arbitrary shaped section with a constant wall thickness and made of a homogeneous material, the torsional stiffness reduces to

$$J = \frac{4Gt\mathcal{A}^2}{l}. \quad (6.76)$$

where $l = \int_C ds$ is the perimeter of contour $\mathcal{C}$.

The same result can be obtained with the help of the membrane analogy developed in section (5.4). Fig. 6.23 shows the thin-walled cross-section and the membrane stretched between the inner and outer contours. The boundary condition (5.67) requires the stress

function to be constant along the contours defining the section. Since the section is bounded by two distinct contours, the stress function has a constant value along one contour, say the outer contour, and a different constant value along the inner contour. This translates into a membrane stretched between inner and outer planar contours with an unknown relative elevation $(p/S)h$. The slope of the membrane is assumed to remain constant in the direction of $\mathbf{n}$ because the distance t between the two contours is very small, i.e.

$$\frac{\partial u_1}{\partial n} = \frac{0 - (p/S)h}{t}. \quad (6.77)$$

The membrane analogy (5.77) then yields the corresponding shear stress

$$\tau_s = 2G\kappa_1 \frac{h}{t}. \quad (6.78)$$

The assumed uniformity of the membrane slope in the normal direction translates into a uniform distribution of shearing stress across the wall thickness, the customary assumption for thin-walled beams. The corresponding shear flow $f = -\tau_s t = 2G\kappa_1 h$ is a constant over the contour as was previously found from the local equilibrium eq. (6.67).

The total torque is related to the volume under the deflected membrane by eq. (5.82). Hence,

$$M_1 = \frac{2G\kappa_1}{p/S} 2V = 2G\kappa_1 2\mathcal{A}h. \quad (6.79)$$

Eliminating h between eqs. (6.78) and (6.79) then yields

$$M_1 = 2\mathcal{A}t\tau_s = 2\mathcal{A}f. \quad (6.80)$$

Finally, since the mid contour is a contour of constant membrane deflection, it follows from eq. (5.81) that

$$\int_C \tau_s \, ds = 2G\kappa_1 \mathcal{A}. \quad (6.81)$$

Introducing the shearing stress from eq. (6.80) leads to

$$\int_C \frac{M_1}{2\mathcal{A}t} \, ds = 2G\kappa_1 \mathcal{A}. \quad (6.82)$$

Solving for the torque in term of the twist rate yields

$$M_1 = \frac{4G\mathcal{A}^2}{\int_C \frac{ds}{t}} \kappa_1. \quad (6.83)$$

Eqs. (6.80) and (6.83) are identical to the previously obtained results, eqs. (6.70) and (6.74), respectively. The only difference is that the membrane analogy assumes a homogeneous section, i.e. $G = \text{constant}$. As a result, the shearing modulus has been factored out of the integral over the contour.

6.6.3 Comparison of open and closed sections

The torsional behavior of closed sections sharply contrasts with that of open sections. The shear stress is *uniformly* distributed through the thickness of the wall of closed sections, whereas a *linear* distribution through the wall thickness is found in open sections. The torsional stiffness is proportional to the square of the area enclosed for a closed section (6.75), in contrast with a thickness to the cube proportionality for open sections (5.92).

To illustrate these sharp differences, consider a thin strip of circular shape, and a thin-walled circular tube, both of identical mean radius R_m and thickness t , as depicted in fig. 6.24. The torsional stiffness of the open and closed sections, denoted J_{open} and J_{closed} , respectively, are given by eqs. (5.92), and (5.24), respectively, as:

$$J_{open} = \frac{1}{3} G 2\pi R_m t^3; \quad J_{closed} = 2\pi G R_m^3 t.$$

Their ratio becomes:

$$\frac{J_{closed}}{J_{open}} = 3 \left(\frac{R_m}{t} \right)^2. \quad (6.84)$$

The maximum shear stresses τ_{open}^{max} , and τ_{closed}^{max} in the open, and closed sections, respectively, are found from eqs. (5.94), and (5.29), respectively, as:

$$\tau_{open}^{max} = \frac{3M_1}{2\pi R_m t^2}; \quad \tau_{closed}^{max} = \frac{M_1}{2\pi R_m^2 t}.$$

Their ratio becomes:

$$\frac{\tau_{open}^{max}}{\tau_{closed}^{max}} = 3 \left(\frac{R_m}{t} \right). \quad (6.85)$$

Consider a thin walled construction where $R_m = 20t$. The torsional stiffness of the closed section will be 1,200 times higher than that of the open section, and the maximum shearing stress in the open section will be 60 times larger than that of the closed section. In other words, the closed section would be able to carry a 60 times larger torque for an equal shearing stress level.

6.6.4 Torsion of combined open and closed sections

In the previous section, the behavior of open and closed sections was shown to contrast sharply. In practical situations, one is often confronted with cross-sections presenting a combination of open and closed paths, as depicted in fig. 6.25. This section consists of the combination of a closed trapezoidal contour and two thin rectangular strips. All components have a constant thickness t , and the other dimensions are shown on the figure. The twist rate is identical for the trapezoidal section and the rectangular strips, whereas the torques they carry, denoted M_{1trap} and M_{1strip} , respectively, are

$$M_{1trap} = J_{trap} \kappa_1; \quad M_{1strip} = J_{strip} \kappa_1. \quad (6.86)$$

The torsional stiffnesses of the trapezoidal section and rectangular strips, denoted J_{trap} and J_{strip} , respectively, are evaluated with the help of (6.76) and (5.92), respectively,

$$J_{trap} = \frac{4Gt\mathcal{A}^2}{l}; \quad J_{strip} = G \frac{dt^3}{3},$$

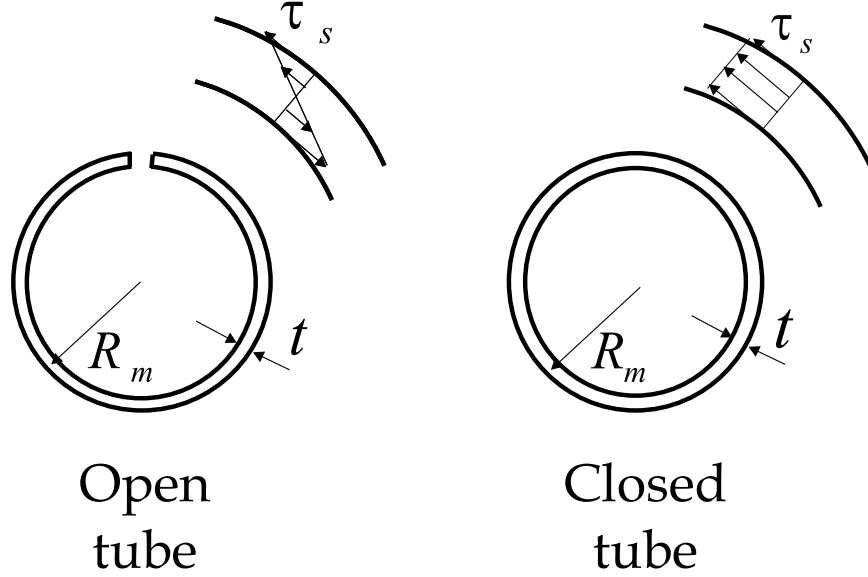


Figure 6.24: A thin-walled open tube and a thin-walled closed tube.

where l is the length of the perimeter of the trapezoidal section and $\mathcal{A}$ the enclosed area. The total torque M_{1t} is the sum of the torques carried by the three components of the section

$$M_{1t} = M_{1\text{trap}} + 2M_{1\text{strip}} = J_{\text{trap}} \left(1 + 2 \frac{J_{\text{strip}}}{J_{\text{trap}}} \right) \kappa_1 = J_{\text{trap}} \left[1 + \frac{2}{3} \frac{dl}{(b_1 + b_2)^2} \left(\frac{t}{h} \right)^2 \right] \kappa_1.$$

For thin-walled sections the last term in the bracket is clearly negligible, i.e.

$$J \approx J_{\text{trap}}$$

In other words, the total torsional stiffness is that of the closed trapezoidal section, the contribution of the open parts of the section is negligible. The twist rate of the section is $\kappa_1 = M_{1t}/J_{\text{trap}}$, and the torques carried by the individual components of the section become

$$M_{1\text{trap}} \approx J_{\text{trap}} \frac{M_{1t}}{J_{\text{trap}}} = M_{1t}; \quad M_{1\text{strip}} \approx \frac{J_{\text{strip}}}{J_{\text{trap}}} M_{1t}.$$

Finally, the maximum shear stress in the trapezoidal section and rectangular strips are found from eq. (6.70) and (5.94), respectively

$$\tau_{\text{trap}}^{\max} = \frac{M_{1\text{trap}}}{2\mathcal{A}t} \approx \frac{1}{2\mathcal{A}t} M_{1t}; \quad \tau_{\text{strip}}^{\max} = \frac{3M_{1\text{strip}}}{dt^2} \approx \frac{3}{dt^2} \frac{J_{\text{strip}}}{J_{\text{trap}}} M_{1t}.$$

The ratio of these stresses is now

$$\frac{\tau_{\text{strip}}^{\max}}{\tau_{\text{trap}}^{\max}} = \frac{l}{b_1 + b_2} \left(\frac{t}{h} \right).$$

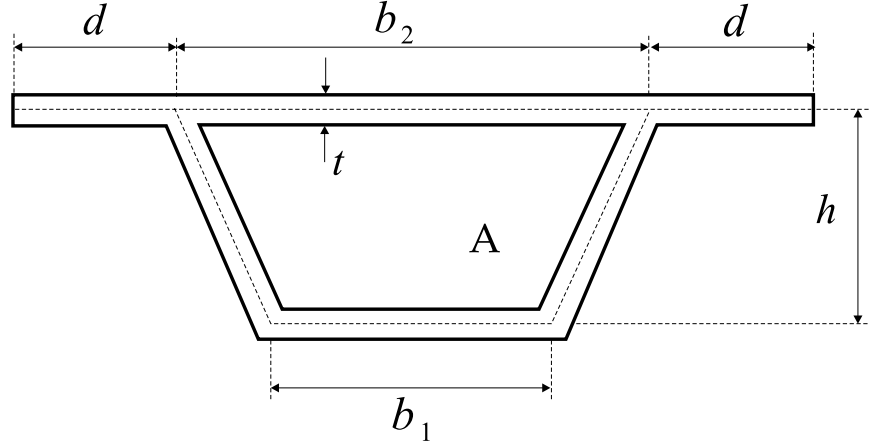


Figure 6.25: A combined open and closed cross-section.

Clearly, the maximum shear stress in the strip is far smaller than that in the trapezoidal section.

In summary, when dealing with a combination of open and closed sections under torsion, the contribution of the open part can be neglected when evaluating the torsional stiffness, and the shear stress in the open section is far smaller than that in the closed section.

6.6.5 Torsion of multi-cellular sections

The membrane analogy will be used once more to investigate the behavior of the multi-cellular section shown in fig. 6.26. For this problem, the membrane must be stretched along three planar contours: the outer contour of the section taken with a zero deflection, and the inner contours of the left and right cells with unknown deflections $(p/S)h_1$ and $(p/S)h_2$, respectively. The constant slope of the membrane in the direction normal to the contours translates into uniform shear stress distributions through the thickness of the wall of each of the three curves $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ found from the membrane analogy (5.79) as

$$\tau_{s1} = 2G\kappa_1 \frac{h_1}{t}, \quad \tau_{s2} = 2G\kappa_1 \frac{h_2}{t}, \quad \tau_{s3} = 2G\kappa_1 \frac{h_2 - h_1}{t} = \tau_{s2} - \tau_{s1}, \quad (6.87)$$

respectively. As a result, the shear flows f_1 , f_2 , and $f_3 = f_2 - f_1$ corresponding to each of the three curves $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively, are constants. Relationship (5.81) is then applied to the inner contour of each cell

$$\int_{\mathcal{C}_1 - \mathcal{C}_3} \tau_s \, ds = 2G\kappa_1 \mathcal{A}_1; \quad \int_{\mathcal{C}_2 + \mathcal{C}_3} \tau_s \, ds = 2G\kappa_1 \mathcal{A}_2. \quad (6.88)$$

Introducing (6.87) then yields the following set of linear equations for the unknown deflections of the membrane

$$\begin{aligned} (\alpha_1 + \alpha_3) h_1 - \alpha_3 h_2 &= \mathcal{A}_1; \\ -\alpha_3 h_1 + (\alpha_2 + \alpha_3) h_2 &= \mathcal{A}_2, \end{aligned} \quad (6.89)$$

where

$$\alpha_1 = \int_{C_1} \frac{ds}{t}; \quad \alpha_2 = \int_{C_2} \frac{ds}{t}; \quad \alpha_3 = \int_{C_3} \frac{ds}{t}. \quad (6.90)$$

The solution of these equations yields the membrane deflection

$$h_1 = \frac{(\alpha_2 + \alpha_3)\mathcal{A}_1 + \alpha_3\mathcal{A}_2}{\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_1\alpha_2}; \quad h_2 = \frac{\alpha_3\mathcal{A}_1 + (\alpha_1 + \alpha_3)\mathcal{A}_2}{\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_1\alpha_2}. \quad (6.91)$$

The torque is given by (5.82) as

$$M_1 = \frac{2G\kappa_1}{p/S} 2V = 2G\kappa_1 (2\mathcal{A}_1 h_1 + 2\mathcal{A}_2 h_2). \quad (6.92)$$

Introducing the membrane deflection yields $M_1 = J\kappa_1$ where the torsional stiffness is

$$J = 4G \frac{\alpha_1\mathcal{A}_2^2 + \alpha_2\mathcal{A}_1^2 + \alpha_3(\mathcal{A}_1 + \mathcal{A}_2)^2}{\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_1\alpha_2}. \quad (6.93)$$

Finally, the stress flows in the various portions of the section are found by introducing eq. (6.91) into eq.(6.87)

$$\begin{aligned} f_1 &= \frac{(\alpha_2 + \alpha_3)\mathcal{A}_1 + \alpha_3\mathcal{A}_2}{\alpha_1\mathcal{A}_2^2 + \alpha_2\mathcal{A}_1^2 + \alpha_3(\mathcal{A}_1 + \mathcal{A}_2)^2} \frac{M_1}{2}; \\ f_2 &= \frac{\alpha_3\mathcal{A}_1 + (\alpha_1 + \alpha_3)\mathcal{A}_2}{\alpha_1\mathcal{A}_2^2 + \alpha_2\mathcal{A}_1^2 + \alpha_3(\mathcal{A}_1 + \mathcal{A}_2)^2} \frac{M_1}{2}; \end{aligned} \quad (6.94)$$

and $f_3 = f_2 - f_1$.

This solution process is readily extended to multi-cellular section with N cells. The membrane analogy leads to constant shear flows in each curve the section is comprised of. Relationship (5.81) is the applied to the inner contours of each of the N cell resulting in a set of N linear equations for the unknown membrane deflections of the contours of each of the N cells. The solution of this system gives the shear flows in the various components of the section. The torque and torsional stiffness are then obtained from (5.82).

6.6.6 Problems

Problem 6.5

The cross-section of a high lift device is shown in fig 6.27. The aerodynamic pressure acting on the lower panel of the device has a net resultant $F_3 = 100 \times 10^3 \text{ N}$ and its line of action is aligned with the left web, as indicated on the figure. Material properties are: $E = 73 \text{ GPa}$, $G = 30 \text{ GPa}$.

1. Find and sketch the distribution of shear stress flow associated with the shear force F_3 .
2. Find and sketch the distribution of shear stress associated with torsion of the section.
3. Find and sketch the total shear stress distribution in the section.
4. Indicate the location and magnitude of the maximum shear stress in the section.

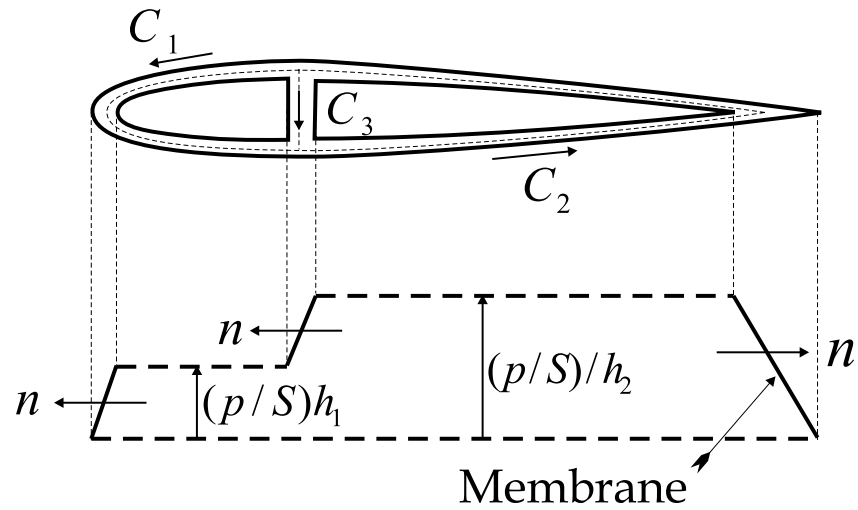


Figure 6.26: A thin-walled, multi-cellular section under torsion.

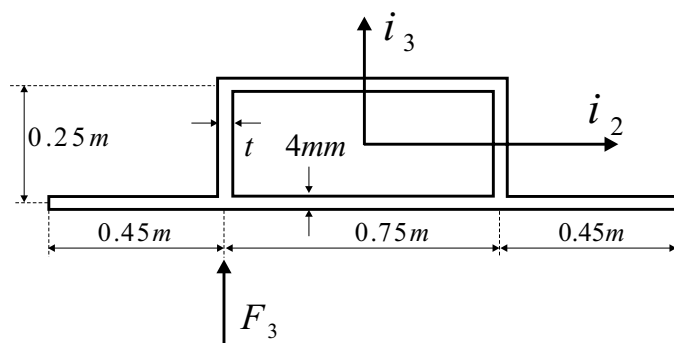


Figure 6.27: High lift device subjected to a transverse shear force.

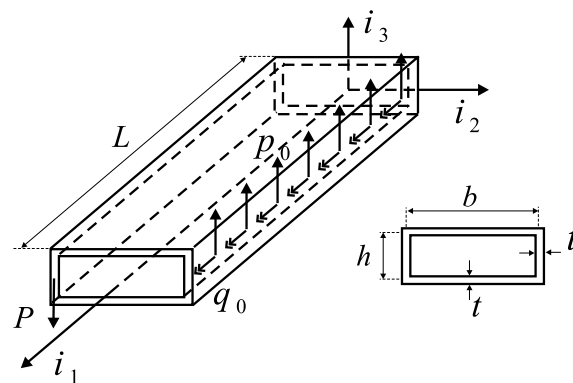


Figure 6.28: Thin-walled, rectangular box beam.

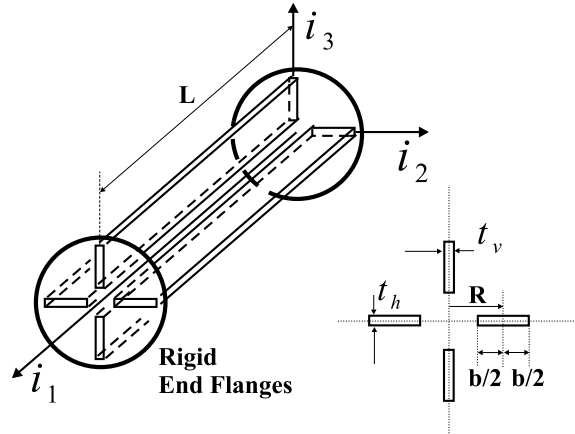


Figure 6.29: Configuration of the four-web flexure.

Problem 6.6

Consider the thin-walled, rectangular box beam of length $L = 2 \text{ m}$ shown in fig. 6.28. The cross-section has a width $b = 0.2 \text{ m}$, a height $h = 0.1 \text{ m}$, and a constant wall thickness $t = 5 \times 10^{-3} \text{ m}$. The cantilevered beam is subjected to a tip load $P = 5 \times 10^3 \text{ N}$ acting in the plane of the tip section as indicated on the figure, a distributed transverse load $p_0 = 20 \times 10^3 \text{ N/m}$, and a distributed torque $q_0 = 1.0 \times 10^3 \text{ N}$.

1. Find the distribution of axial stress in the root section of the beam. Find the magnitude and location of the maximum axial stress. [10 points]
2. Find the distribution of shear stress at the root of the beam due to the transverse shearing force. [10 points]
3. Find the distribution of shear stress at the root of the beam due to the torsional loads. [10 points]

Problem 6.7

Fig. 6.29 depicts a flexure composed two rigid circular flanges and four flexible webs (homogeneous material of Young's modulus E and shearing modulus G). The flexure is subjected to a tip axial load P_1 , a tip torque Q_1 , and tip bending moments Q_2 and Q_3 . The resulting tip axial displacement is $U_1(L)$, tip rotation $\phi_1(L)$, $\phi_2(L)$ and $\phi_3(L)$. Find

1. The shear center of the section.
2. The axial stiffness S_a of the flexure: $S_a = P_1/U_1(L)$. Give the nondimensional stiffness LS_a/EbR .
3. The torsional stiffness: $S_1 = Q_1/\phi_1(L)$. Give the nondimensional stiffness LS_1/EbR^3 .
4. The bending stiffnesses: $S_2 = Q_2/\phi_2(L)$ and $S_3 = Q_3/\phi_3(L)$. Give the nondimensional stiffnesses LS_2/EbR^3 and LS_3/EbR^3 .

Use an appropriate approximation of the torsional stiffness of individual webs.

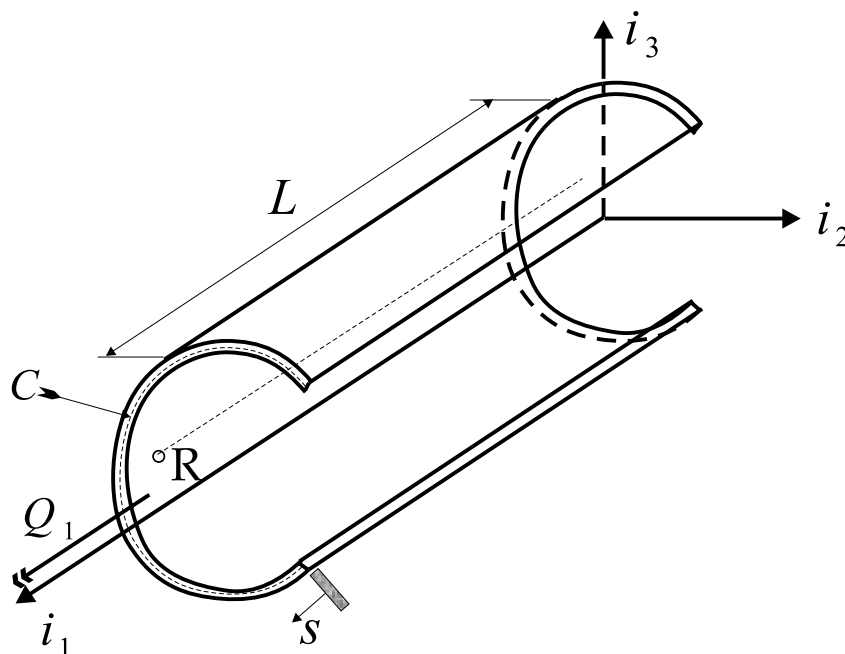


Figure 6.30: Thin-walled beam subjected to an applied torque.

6.7 Warping of thin-walled beams under torsion

When a thin-walled beam is subjected to an applied torque, shearing stresses are generated as discussed in the previous section. In turns, these shearing stresses cause out-of-plane deformations of the cross-section called *warping*. These deformation can significantly alter the torsional behavior of beams, specially in the presence of open sections.

Consider a thin-walled beam subjected to pure torsional moments as depicted in fig. 6.30. Here again, the formulation is simplified by selecting the axes as the principal centroidal axes of bending. Under the effect of the applied torque, the beam rotates about a point R , called the *center of twist*, with coordinates (x_{2r}, x_{3r}) . The location of the center of twist is as yet unknown.

The analysis of this problem starts with an assumed displacement field similar to that of the Saint-Venant solution described in section (5.3.1). The axial displacement component is proportional to the twist rate and characterized by an unknown warping function Ψ , whereas the in-plane displacement components describe a rigid body rotation of the section about the center of twist

$$\begin{aligned} u_1(x_1, s) &= \Psi(s) \kappa_1(x_1); \\ u_2(x_1, s) &= -(x_3 - x_{3r}) \Phi_1(x_1); \quad u_3(x_1, s) = (x_2 - x_{2r}) \Phi_1(x_1). \end{aligned} \quad (6.95)$$

In view of the thin wall assumption, the value of the warping function is assumed to remain constant across the wall thickness and becomes a sole function of the curvilinear variable s .

The strain field follows from this assumed displacement field

$$\varepsilon_1 = \Psi(s) \frac{d\kappa_1}{dx_1}; \quad (6.96)$$

$$\varepsilon_2 = 0; \quad \varepsilon_3 = 0; \quad \gamma_{23} = 0; \quad (6.97)$$

$$\gamma_{12} = \left[\frac{d\Psi}{dx_2} - (x_3 - x_{3r}) \right] \kappa_1; \quad \gamma_{13} = \left[\frac{d\Psi}{dx_3} + (x_2 - x_{2r}) \right] \kappa_1. \quad (6.98)$$

In the present analysis, the beam is undergoing *non-uniform torsion*, i.e. $d\kappa_1/dx_1 \neq 0$. This contrasts with the Saint-Venant solution (5.3.1), and the analysis developed in section (6.6) where uniform torsion was assumed. As a result, the axial strain (6.96) does not vanish. The in-plane strain components (6.97) vanish because of the assumed rigid body rotation of the section. The shearing strain components (6.98) depend on the partial derivatives of the warping function and are proportional to the twist rate.

The non vanishing components of the stress field are readily obtained

$$\sigma_1 = E\Psi(s) \frac{d\kappa_1}{dx_1}; \quad (6.99)$$

$$\tau_{12} = G \left[\frac{d\Psi}{dx_2} - (x_3 - x_{3r}) \right] \kappa_1; \quad \tau_{13} = G \left[\frac{d\Psi}{dx_3} + (x_2 - x_{2r}) \right] \kappa_1. \quad (6.100)$$

For thin-walled beams, the only non vanishing component of shearing stress is the component τ_s tangent to $\mathcal{C}$, as discussed in section (6.1.2). For eq. (6.4), this shearing stress component writes

$$\tau_s = G \left[\frac{\partial\Psi}{\partial x_2} \frac{dx_2}{ds} + \frac{\partial\Psi}{\partial x_3} \frac{dx_3}{ds} + (x_2 - x_{2r}) \frac{dx_3}{ds} - (x_3 - x_{3r}) \frac{dx_2}{ds} \right] \kappa_1. \quad (6.101)$$

The first two terms represent the total derivative of the warping function with respect to s , whereas the last two terms are the distance from the twist center to the tangent to the contour, denoted r_r (see eq. (6.14)). Hence

$$\tau_s = G \left(\frac{d\Psi}{ds} + r_r \right) \kappa_1. \quad (6.102)$$

To complete this analysis and determine the warping function, a distinction must now be made between open and closed sections.

6.7.1 Warping of open sections

The torsional behavior of thin-walled open section was investigated in section (6.6.1) with the help of the membrane analogy. The shear stress distribution was found to be linearly distributed across the wall thickness. As a result, the shear stress τ_s vanishes along $\mathcal{C}$. Equation (6.102) then implies

$$\tau_s = G \left(\frac{d\Psi}{ds} + r_r \right) \kappa_1 = 0. \quad (6.103)$$

Hence, the warping function satisfies the following differential equation

$$\frac{d\Psi}{ds} = -r_r = - \left(r_o - x_{2r} \frac{dx_3}{ds} + x_{3r} \frac{dx_2}{ds} \right). \quad (6.104)$$

To integrate this equation and determine the warping function, we first define a function $\Gamma(s)$ such that

$$\frac{d\Gamma}{ds} = -r_o. \quad (6.105)$$

An arbitrary boundary condition is used to evaluate $\Gamma(s)$. Integration of (6.104) then yields the warping function

$$\Psi(s) = \Gamma(s) + x_{2r} x_3 - x_{3r} x_2 + a, \quad (6.106)$$

where a is an integration constant. To fully define the warping function, it is necessary to evaluate this integration constant, as well as the coordinates of the center of twist.

In the case of uniform torsion $\kappa_1 = \text{constant}$, the warping of all sections is identical (see eq. (6.95)), and the axial strains, eq. (6.96), and stresses, eq. (6.99) both vanish. The integration constant and the location of the center of twist cannot be determined. The indeterminate part of the warping function $x_{2r} x_3 - x_{3r} x_2 + a$ represents a rigid body displacement field that does not affect the state of strain or stress in the bar. In fact, any point can be selected as the center of twist. However, most practical problems involve non-uniform torsion because the applied torque varies along the axis of the beam, or warping is constrained at some point along the span. In that case, two neighboring sections warp a different amount giving rise to axial strains which in turns, generate axial stresses, though the beam is acted upon solely by a torque. As a result, though the axial stress does not vanish at all points of the section, the axial force, eq. (6.8), and bending moments, eq. (6.9) must vanish, since no such loads were applied.

The vanishing of the axial force implies

$$F_1 = \int_C \sigma_1 t ds = 0. \quad (6.107)$$

Introducing the axial stress (6.99), and the warping function (6.106) yields

$$\int_C E\Gamma t ds + x_{2r} \int_C E x_3 t ds - x_{3r} \int_C E x_2 t ds + a \int_C E t ds = 0. \quad (6.108)$$

The second and third integral of this expression are zero because the origin of the axes was selected at the centroid. The last integral represents the axial stiffness S , see equation (3.18). The integration constant is found as

$$a = -\frac{1}{S} \int_C E\Gamma t ds. \quad (6.109)$$

Next, the vanishing of the bending moments defined by eq. (6.9) implies

$$\int_C E\Gamma x_3 t ds + x_{2r} \int_C E x_3^2 t ds - x_{3r} \int_C E x_2 x_3 t ds + a \int_C E x_3 t ds = 0. \quad (6.110)$$

The second integral of this expression is the bending stiffness I_{22} , the third the cross-bending stiffness I_{23} which is zero because the axes are principal axes of bending, and the last integral also vanishes because the axes origin is at the centroid. The coordinate x_{2r} of the center of twist follows

$$x_{2r} = -\frac{1}{I_{22}} \int_C E\Gamma x_3 t ds. \quad (6.111)$$

Finally, enforcing the bending moment M_3 to be zero leads to

$$x_{3r} = \frac{1}{I_{33}} \int_C E \Gamma x_2 t ds. \quad (6.112)$$

The procedure to compute the warping function for thin-walled beams with an open section consists of the following steps.

1. Compute the location of the centroid and the orientation of the principal axes of bending of the section. Set the axes accordingly;
2. Compute the function $\Gamma(s)$ by integration eq. (6.105). Use an arbitrary boundary condition;
3. Compute the integration constant a with the help of eq. (6.109);
4. Compute the location of the center of twist from eqs. (6.111) and (6.112);
5. The warping function is then fully defined by eq. (6.106).

The shape of the warping function describes the out-of-plane deformation of the cross-section. Indeed, for (6.95), the warping function gives the distribution of axial displacement within a scaling factor κ_1 . For non-uniform torsion, the warping function also describes the distribution of axial strain (6.96) over the section. Here the warping function gives the distribution of axial strain with a scaling factor $d\kappa_1/dx_1$, the rate of change of the twist rate along the span. Finally, for sections made of a homogeneous material, i.e. $E(s) = \text{constant}$, the axial stress (6.99) is also distributed according to the warping function, this time with a scaling factor $E d\kappa_1/dx_1$.

It is worthwhile noticing that the vanishing of the axial stress resultants imply the following properties of the warping function

$$\int_C E \Psi t ds = \int_C E \Psi x_2 t ds = \int_C E \Psi x_3 t ds = 0. \quad (6.113)$$

Example: Warping of a C-channel

The determination of the warping function will be illustrated with the example of the C-channel depicted in fig. 6.11. The first step is to compute the function $\Gamma(s)$ from eq. (6.105). Starting with the lower flange where $r_o = -h/2$, we find

$$\Gamma(s) = \frac{hs}{2} + c_1,$$

where c_1 is an integration constant evaluated with the help of an arbitrary boundary condition, say $\Gamma(s) = 0$ at $s = 0$. The function Γ becomes

$$\Gamma_1 = \frac{hs}{2}.$$

A similar process is used for the web and upper flange, where $r_o = -d$ and $r_o = -h/2$, respectively. Continuity of the function Γ is enforced to find

$$\Gamma_2 = ds + \frac{h}{2}(b + d); \quad \Gamma_3 = \frac{hs}{2} + \frac{h}{2}(b + 2d).$$

The next step is to evaluate the integration constant with the help of eq. (6.109)

$$a = -\frac{Et}{S} \left[\int_0^b \Gamma_1 ds + \int_{-h/2}^{+h/2} \Gamma_2 ds + \int_0^b \Gamma_3 ds \right] = -\frac{h}{2}(b + d),$$

and the coordinates of the center of twist from eqs. (6.111) and (6.112)

$$x_{2r} = -\frac{Et}{I_{22}} \left[\int_0^b \Gamma_1 \left(-\frac{h}{2}\right) ds + \int_{-h/2}^{+h/2} \Gamma_2 s ds + \int_0^b \Gamma_3 \frac{h}{2} ds \right] = -d - \frac{h^2 b^2}{4I};$$

$$x_{3r} = \frac{Et}{I_{33}} \left[\int_0^b \Gamma_1 (b - d - s) ds + \int_{-h/2}^{+h/2} \Gamma_2 (-d) ds + \int_0^b \Gamma_3 (s - d) ds \right] = 0.$$

This last result could have been readily obtained from the symmetry of the problem. The warping function then follows from eq. (6.106)

$$\Psi_1 = \frac{h}{2}(s + e - b); \quad \Psi_2 = -es; \quad \Psi_3 = \frac{h}{2}(s - e), \quad (6.114)$$

where $e = h^2 b^2 / 4I$. Fig. 6.31 shows the warping function and the location of the center of twist. Clearly, this center of twist is at a distance e from the web. It is interesting to note that the shear center, found in section (6.3.1) and the center of twist coincide for this cross-section.

6.7.2 Warping of closed sections

The torsional behavior of thin-walled closed sections was investigated in section (6.6.2). The shearing stress was assumed to be uniformly distributed across the wall thickness, and found to be proportional to the area $\mathcal{A}$ enclosed by $\mathcal{C}$, see eq. (6.70)

$$\tau_s = \frac{J\kappa_1}{2\mathcal{A}t}. \quad (6.115)$$

On the other hand, this shear stress is related to the warping function by eq. (6.102). Hence,

$$\frac{d\Psi}{ds} = \frac{J}{2\mathcal{A}Gt} - r_r. \quad (6.116)$$

This is the governing equation for the warping function, and it should be compared to eq. (6.104) that applies for open sections. The process of integration of eq. (6.116) closely follows that used for open sections. First, a function Γ is defined

$$\frac{d\Gamma}{ds} = \frac{J}{2\mathcal{A}Gt} - r_o. \quad (6.117)$$

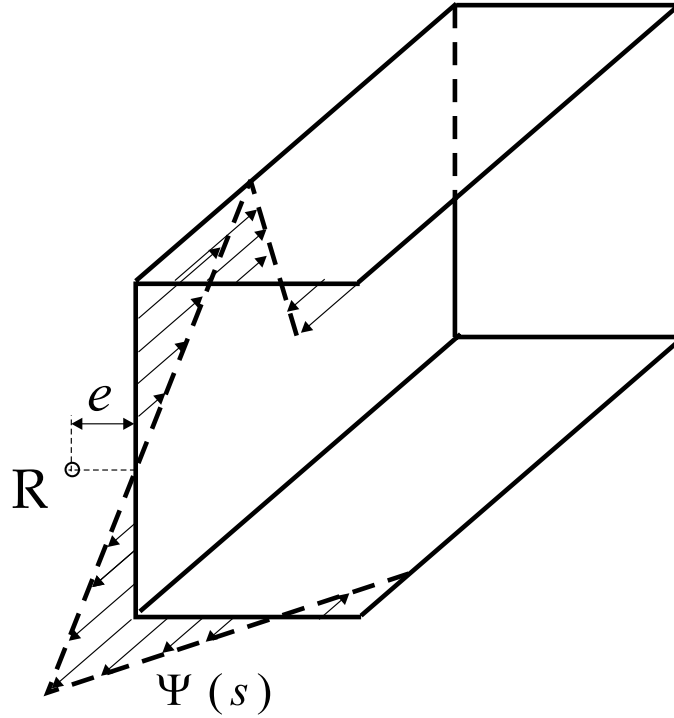


Figure 6.31: The warping function for a C-channel.

Here again, an arbitrary boundary condition is used to evaluate Γ . Integrating eq. (6.116) yields the warping function in the form of eq. (6.106). The integration constant a , and the location of the center of twist are then found by enforcing the vanishing of the axial force, and bending moments, respectively.

In summary, the procedure for the determination of the warping function of closed sections is identical to that for open sections, except that the governing equation for the function Γ is (6.117) rather than (6.105).

Example: Warping function for a rectangular section

The procedure is illustrated with the example of the thin-walled rectangular beam shown in fig. 6.32. The width and height of the section are $2a$ and $2b$, respectively, and the thickness t is uniform. The torsional stiffness is found from eq. (6.75) so that

$$\frac{J}{2AGt} = \frac{2\mathcal{A}}{t \int_c \frac{ds}{t}} = \frac{2ab}{a+b}.$$

The governing equation for Γ becomes

$$\frac{d\Gamma}{ds} = \frac{2ab}{a+b} - r_o,$$

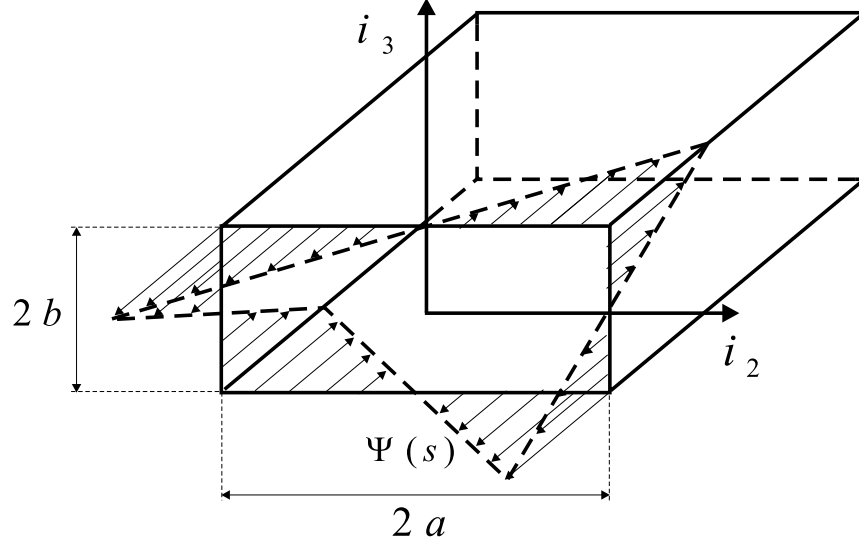


Figure 6.32: Thin-walled beam with a rectangular cross-section.

and integration yields

$$\Gamma_1 = \frac{a-b}{a+b} bs; \quad \Gamma_2 = -\frac{a-b}{a+b} as; \quad \Gamma_3 = \frac{a-b}{a+b} bs; \quad \Gamma_4 = -\frac{a-b}{a+b} as.$$

For the upper flange the boundary condition $\Gamma_1 = 0$ at $s = 0$ was used, and continuity was enforced for the other components of the section. For obvious symmetry reasons $a = x_{2r} = x_{3r} = 0$, as can be verified with the help of eqs. (6.109), (6.111), and (6.112). The warping function $\Psi = \Gamma$ now becomes

$$\Psi_1 = \frac{a-b}{a+b} bs; \quad \Psi_2 = -\frac{a-b}{a+b} as; \quad \Psi_3 = \frac{a-b}{a+b} bs; \quad \Psi_4 = -\frac{a-b}{a+b} as. \quad (6.118)$$

and is depicted in fig. 6.32. For non-uniform torsion, this warping function also represents the distribution of axial strain and axial stress, to suitable scales.

6.7.3 Warping of multi-cellular sections

In the case of multi-cellular sections the shear flow distribution $f_t(s)$ due to an applied torque must be computed first with the help of the procedure described in section (6.6.5). This shear flow distribution is proportional to the twist rate, i.e. $f_t(s) = \mathcal{G}_t(s) \kappa_1$, and the corresponding shear stress is $\tau_s = \mathcal{G}_t(s)/t$. Comparing this result with eq. (6.102) yields the governing equation for the warping function

$$\frac{d\Psi}{ds} = \frac{\mathcal{G}_t(s)}{Gt} - r_r. \quad (6.119)$$

The procedure for the determination of the warping function of multi-cellular sections then exactly mirrors that for open and closed sections, except that the governing differential

equation for the function Γ is now

$$\frac{d\Gamma}{ds} = \frac{\mathcal{G}_t(s)}{Gt} - r_o. \quad (6.120)$$

6.8 Equivalence of the shear and twist centers

The analysis of the shear flow distribution in thin-walled beams under shear forces led to the concept of shear center. The shear center was defined as the point at which transverse shearing forces must be applied for the beam to bend without twisting. This important concept allowed the decoupling of bending and twisting problems, as discussed in section (6.5). On the other hand, a center of twist was introduced for the analysis of thin-walled beam under torsion.

We now propose to show that the shear and twist centers coincide for open sections. Consider eq. (6.60) for the coordinate x_{2k} of the shear center, and introduce the function Γ defined by eq. (6.105)

$$x_{2k} = \int_C \mathcal{G}_3 r_o ds = - \int_C \mathcal{G}_3 \frac{d\Gamma}{ds} ds. \quad (6.121)$$

Integrating by parts then yields

$$x_{2k} = \int_C \Gamma \frac{d\mathcal{G}_3}{ds} ds - [\mathcal{G}_3 \Gamma]_{\text{boundary}}. \quad (6.122)$$

The boundary term vanishes since $\mathcal{G}_3 = 0$ at the boundaries. Next, the governing equation (6.50) for $\mathcal{G}_3$ is introduced

$$x_{2k} = - \int_C \frac{Et}{I_{22}} x_3 \Gamma ds = - \frac{1}{I_{22}} \int_C E \Gamma x_3 t ds = x_{2r}, \quad (6.123)$$

where the last equality follows from eq. (6.111). A similar reasoning leads to $x_{3k} = x_{3r}$, establishing the equivalence of the shear and twist centers for open sections. The equivalence also holds for closed section, as can be shown by a similar development. In fact, this equivalence is a direct consequence of Betti's reciprocal theorem.

6.9 Non-uniform torsion

A thin-walled beam under non-uniform torsion is subjected to a complex state of stress that involves both shearing stresses and axial stresses generated by differential warping. This axial stress is uniform across the thickness, and the associated axial flow is denoted $n_w = t\sigma_1$. As discussed in section (6.7.1), though the axial stress does not vanish at all points of the section, the resulting axial force and bending moment do vanish. This condition implies the global equilibrium of the section, but the local equilibrium equation, eq. (6.16), is not necessarily satisfied. For this local equilibrium to hold, a shear flow f_w called the *warping shear flow* is generated so that

$$\frac{\partial n_w}{\partial x_1} + \frac{\partial f_w}{\partial s} = 0. \quad (6.124)$$

The implication of this new shear flow f_w is investigated for the case of open sections. Introducing expression (6.99) for the axial stress and solving for f_w yields

$$\frac{\partial f_w}{\partial s} = -Et\Psi \frac{d^2\kappa_1}{dx_1^2}. \quad (6.125)$$

The procedure described in section (6.3) would apply for the determination of the warping shear flow f_w depicted in fig. 6.33 for the case of a C-channel.

In the presence of this new shear flow, the question of overall equilibrium arises once more: does the shear flow f_w generate resultant transverse shear forces? The shear force resultant F_{2w} associated with the warping shear flow is evaluated with the help of (6.10)

$$F_{2w} = \int_C f_w \frac{dx_2}{ds} ds = - \int_C x_2 \frac{\partial f_w}{\partial s} ds + [x_2 f_w]_{\text{boundary}} \quad (6.126)$$

The boundary term resulting from the integration by parts vanishes since $f_w = 0$ at the edges of the contour. Introducing eq. (6.125) then yields

$$F_{2w} = \frac{d^2\kappa_1}{dx_1^2} \int_C E\Psi x_2 t ds = 0, \quad (6.127)$$

where the last equality stems from property (6.113) of the warping function. It can be shown in a similar way that $F_{3w} = 0$.

Next, the torque resultant about the shear center generated by the warping shear flow f_w is evaluated with the help of (6.13)

$$M_{1wK} = \int_C f_w r_k ds = - \int_C f_w \frac{d\Psi}{ds} ds, \quad (6.128)$$

where the governing equation (6.104) for the warping function was introduced. Integrating by parts then yields

$$M_{1wK} = \int_C \Psi \frac{df_w}{ds} ds - [f_w \Psi]_{\text{boundary}}, \quad (6.129)$$

where the boundary term vanishes once more. Finally, we introduce eq. (6.125) to find

$$M_{1wK} = -J_w \frac{d^2\kappa_1}{dx_1^2}, \quad (6.130)$$

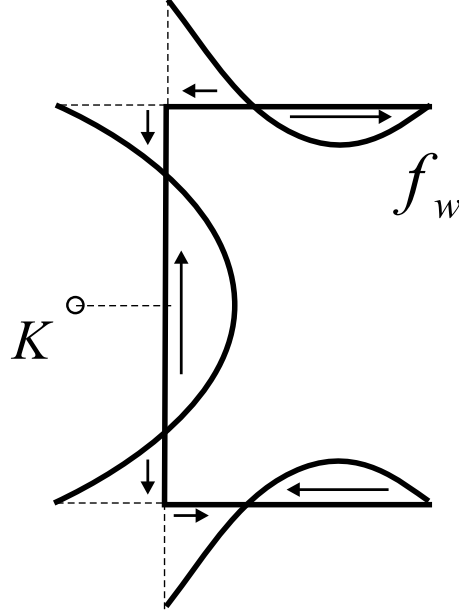
where

$$J_w = \int_C E\Psi^2 t ds, \quad (6.131)$$

is called the *warping stiffness*. The total torque is the sum of that generated by the twist rate and that due to warping, i.e.

$$M_{1K} = J \kappa_1 - J_w \frac{d^2\kappa_1}{dx_1^2}. \quad (6.132)$$

The first component of torque $J \kappa_1$ is that generated by the shear stress distribution described in section (6.6.1), $J_w d^2\kappa_1/dx_1^2$ is an additional contribution due to the warping shear flow f_w . Note that the second contribution vanishes for the case of uniform torsion.

Figure 6.33: Shearing flow f_w for a C-channel.

The equilibrium equation for a differential element of the beam was obtained in section (5.16). Introducing the torque expression (6.132) yields the governing equation for beams undergoing non uniform torsion

$$\frac{d}{dx_1} \left(J \frac{d\Phi_1}{dx_1} - J_w \frac{d^3\Phi_1}{dx_1^3} \right) = -q_1. \quad (6.133)$$

This fourth order equation can be solved to find the beam twist given the applied distributed torque q_1 .

Example: Torsion of a cantilevered beam with free root warping

Consider a uniform cantilevered beam of length L subjected to a tip torque Q . At first, the root condition is such that no twisting is allowed, but warping is free to occur. This condition could be obtained by clamping the beam between two rollers so as to prevent any root rotation without constraining axial displacements. The beam has uniform properties along its length. Hence, the governing equation (6.133) becomes

$$J \frac{d^2\Phi_1}{dx_1^2} - J_w \frac{d^4\Phi_1}{dx_1^4} = 0.$$

At the root of the beam, no twist occurs, i.e. $\Phi_1 = 0$. Since warping is free at the root, the axial stress must vanish, and, in view of (6.99), this implies $d^2\Phi_1/dx_1^2 = 0$. At the tip of the beam, the torque must equal the applied torque Q , and the axial stress must vanish once again. For (6.132) the first condition implies $Q = J d\Phi_1/dx_1 - J_w d^3\Phi_1/dx_1^3$, and the second condition again implies $d^2\Phi_1/dx_1^2 = 0$.

To ease the solution of this problem, a non-dimensional span-wise variable $\xi = x_1/L$ is introduced. The governing equation becomes

$$\Phi_1'''' - k^2 \Phi_1'' = 0, \quad (6.134)$$

and the boundary conditions are

$$\begin{aligned} @ \xi = 0 : \quad & \Phi_1 = 0; \quad \Phi_1'' = 0; \\ @ \xi = L : \quad & \Phi_1'' = 0; \quad k^2 \Phi_1' - \Phi_1''' = \frac{QL^3}{J_w}; \end{aligned}$$

where the notation $(.)'$ is used to denote a derivative with respect to ξ , and

$$k^2 = \frac{JL^2}{J_w}, \quad (6.135)$$

is a non-dimensional parameter that characterizes the importance of the torsional warping stiffness.

The general solution of this equation is

$$\Phi_1 = C_1 + C_2\xi + C_3 \cosh k\xi + C_4 \sinh k\xi, \quad (6.136)$$

where C_1 , C_2 , C_3 , and C_4 are four integration constants. The boundary conditions at the root are used to evaluate C_1 and C_3 to find

$$\Phi_1 = C_2\xi + C_4 \sinh k\xi.$$

The remaining two integration constants are found with the help of the boundary conditions at the tip of the beam. Hence,

$$\Phi_1 = \frac{QL}{J} \xi. \quad (6.137)$$

This solution represents the uniform torsion solution. Indeed, it implies a twist rate $\kappa_1 = d\Phi_1/dx_1 = Q/L = \text{constant}$. The torsional warping stiffness J_w has disappeared from the solution which could have been obtained from the governing equation for uniform torsion developed in chapter 5

Example: Torsion of a cantilevered beam with constrained warping

The cantilevered beam of the above example is considered once again, but the root section is now solidly clamped so as to prevent any warping at the root. At this built-in end, no twisting occurs, i.e. $\Phi_1 = 0$, and no axial displacement is allowed. In view of (6.95), this last condition implies $\kappa_1 = d\Phi_1/dx_1 = 0$. The governing equation of the problem is (6.134) once again, but the boundary conditions now become

$$\begin{aligned} @ \xi = 0 : \quad & \Phi_1 = 0; \quad \Phi_1' = 0; \\ @ \xi = L : \quad & \Phi_1'' = 0; \quad k^2 \Phi_1' - \Phi_1''' = \frac{QL^3}{J_w}; \end{aligned}$$

The general solution of the governing equation still has the form of eq. (6.136), and the boundary conditions at the root are used to evaluate C_3 and C_2 . Hence,

$$\Phi_1 = C_1(1 - \cosh k\xi) + C_4(\sinh k\xi - k\xi).$$

The boundary conditions at the tip allow the evaluation of the remaining integration constants to find

$$\Phi_1 = \frac{QL}{J} \left[\xi - \frac{\sinh k - \sinh k(1 - \xi)}{k \cosh k} \right].$$

The first term represents the linear distribution of twist along the span of the beam characteristic of uniform torsion, as was found in the previous example. The second term represents the influence of the non-uniform torsion induced by the root warping constraint. This constraint decreases the twist of the beam.

Two types of sections will be considered here, the closed rectangular section shown in fig. 6.32, and the open C-channel of fig. 6.31. The warping stiffness of the rectangular section is found by introducing eq. (6.118) into eq. (6.131) and integrating

$$J_w = E \frac{4}{3} \frac{a^2 b^2 t (a - b)^2}{a + b}.$$

The torsional stiffness follows from (6.75)

$$J = G \frac{16a^2 b^2 t}{a + b}$$

The coefficient k defined in (6.135) then becomes

$$k^2 = \left(\frac{G}{E} \right) \left(\frac{2L}{a - b} \right)^2.$$

Note that for $a = b$, the thin walled rectangular section becomes square, the warping function vanishes, as does the warping stiffness, and the uniform torsion solution is recovered. Consider a rectangular section made of aluminum, and for which $a = 4b$. The ratio G/E was found to be $1/2(1 + \nu)$ for isotropic, linear elastic materials, eq. (1.90). The coefficient k then becomes proportional to the aspect ratio L/a . For a long beam ($L/a = 10$), $k = 16.54$, whereas for a shorter beam ($L/a = 5$), $k = 8.27$. On the other hand, if the beam is made of a unidirectional composite for which $G/E = 1/28$, the coefficient k becomes 5.04 and 2.52, for the long and short beams, respectively. Fig. 6.34 shows the twist distribution for the four cases considered.

For the C-channel, the warping stiffness is found from eq. (6.114)

$$J_w = E \frac{h^2 b^3 t}{12} \frac{3b + 2h}{6b + h},$$

and the torsional stiffness was computed in eq. (5.95) as

$$J = \frac{Gt^3}{3} (2b + h).$$

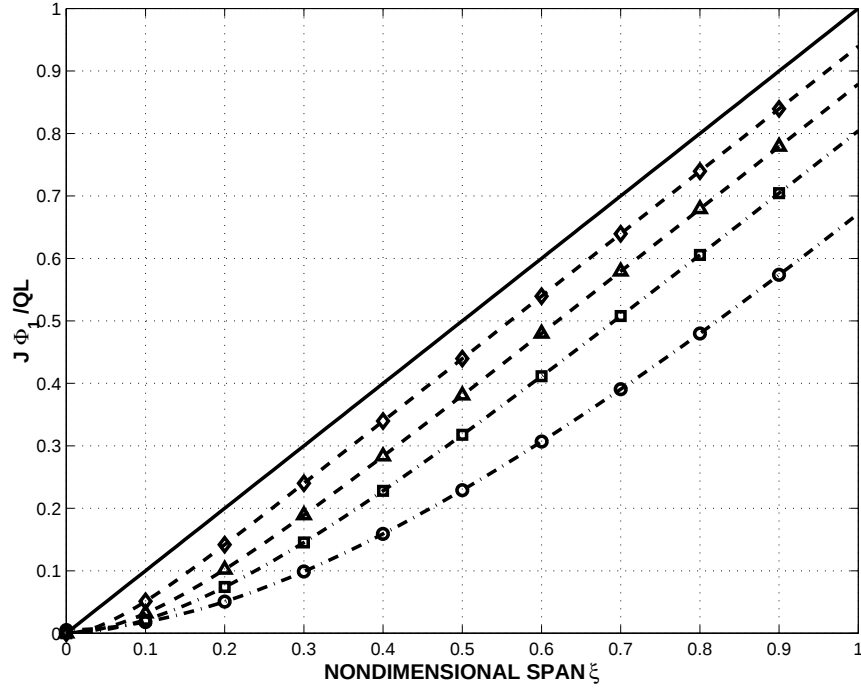


Figure 6.34: Twist distribution for the rectangular section under non-uniform torsion. $k = 16.54$ ($\diamond$), $k = 8.27$ ($\triangle$), $k = 5.04$ ($\square$), $k = 2.52$ ($\circ$).

Finally, the coefficient k becomes

$$k^2 = \left(\frac{G}{E}\right) \left(\frac{L}{h}\right)^2 \frac{2ht^2}{b^3} \frac{(1 + 2b/h)(1 + 6b/h)}{1 + 3b/2h}.$$

Let the C-channel be such that $h = 2b$ and $t = b/10$. If the C-channel is made of aluminum, $k = 2.65$ or 1.33 for the aspect ratios of $L/h = 10$, or 5 , respectively. If the beam is made of a unidirectional composite, the corresponding values are $k = 0.808$ and 0.404 , respectively. Fig. 6.35 shows the twist distribution for the four cases.

It is clear that the effect of the constrained warping becomes increasingly significant as k decreases. The coefficient k is proportional to $\sqrt{G/E}$. For most homogeneous, isotropic metal, Poisson's ratio ν is equal to 0.3 , implying $\sqrt{G/E} = \sqrt{1/2(1 + \nu)} = 0.62$. However, this ratio can be far smaller for highly anisotropic materials. For instance, in the extreme cases of a unidirectional graphite/epoxy material $\sqrt{G/E} = 0.189$. The coefficient k is also proportional to the aspect ratio L/d , where d is a representative dimension of the cross-section; lower values of k will be associated with shorter beams. It is also important to note that *open* sections yield much smaller values of k than *closed* section.

In summary, the importance of non-uniform torsion is characterized by the coefficient k , which itself depends on three factors. First, there is a material effect: the ratio of the shearing to Young's modulus. The second effect is a geometric effect: the aspect ratio of the beam. The last factor is the geometric nature of the cross-section that can be open or closed. The non-uniform torsion effects tend to become important for beams with the

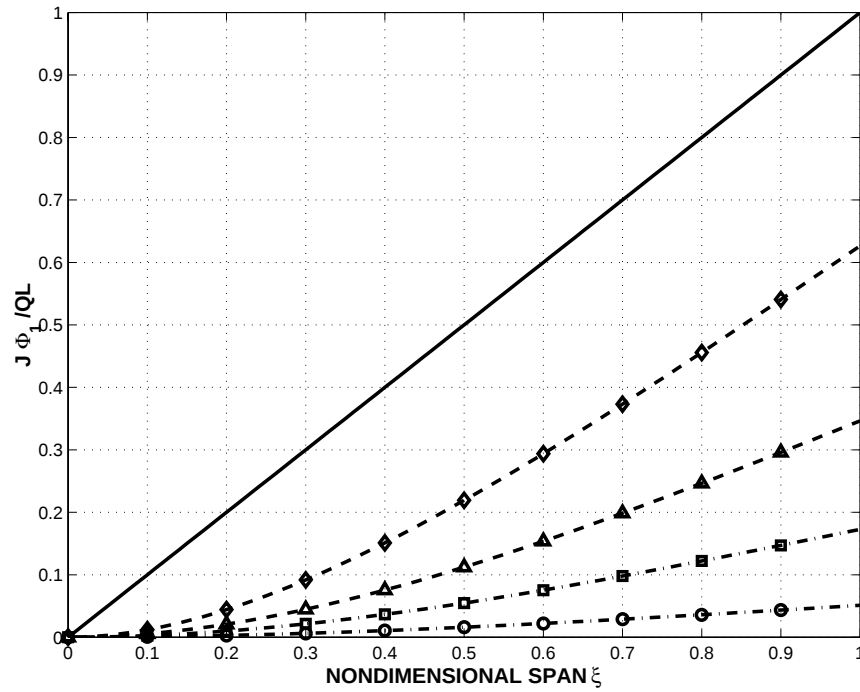


Figure 6.35: Twist distribution for the rectangular section under non-uniform torsion. $k = 2.65$ ($\diamond$), $k = 1.33$ ($\triangle$), $k = 0.808$ ($\square$), $k = 0.404$ ($\circ$).

following characteristics: beams made of a material presenting a low shearing to Young's modulus ratio, beams of lower aspect ratio, and beams with open cross-sections.

6.9.1 Problems

Problem 6.8

Consider the thin-walled beam with a rectangular cross section depicted in fig. 6.36. The beam is clamped at the root and subjected to a tip torque Q_1 .

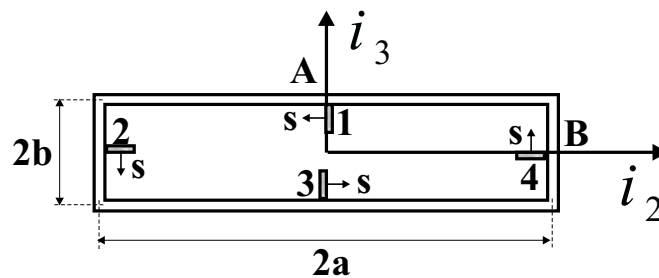


Figure 6.36: Rectangular cross-section of a thin-walled beam.

1. Use the classical theory to solve this problem. Use the following assumed displacement field

$$u_1(x_1, x_2, x_3) = 0;$$

$$u_2(x_1, x_2, x_3) = -x_3 \Phi_1(x_1); \quad u_3(x_1, x_2, x_3) = x_2 \Phi_1(x_1).$$

2. Develop a beam theory for torsion based on the following assumed displacement field

$$u_1(x_1, x_2, x_3) = \Psi(s) \alpha(x_1);$$

$$u_2(x_1, x_2, x_3) = -x_3 \Phi_1(x_1); \quad u_3(x_1, x_2, x_3) = x_2 \Phi_1(x_1),$$

where $\Psi(s)$ is the warping function obtained from thin-walled beam theory, and $\alpha(x_1)$ an unknown function. The warping function is given as

$$\Psi_1(s) = \frac{a-b}{a+b} bs; \quad \Psi_2(s) = -\frac{a-b}{a+b} as; \quad \Psi_3(s) = \frac{a-b}{a+b} bs; \quad \Psi_4(s) = -\frac{a-b}{a+b} as.$$

3. Plot the twist distribution along the span of the beam for the two theories.
4. Plot the shear flow distribution over the cross-section of the beam at $x_1 = 0$ and L , for the two theories.
5. Plot the axial flow distribution over the cross-section of the beam at $x_1 = 0$ and $L/2$, for the two theories.

Use the following parameters:

$$a = 4b; \quad L = 6a; \quad E/G = 20; \quad n = Et\varepsilon_{11}; \quad f = Gt\gamma.$$

Chapter 7

Variational and Energy Principles

7.1 Mathematical Preliminaries

The basic equations of elasticity developed in chapter 1 use the formalism of partial differential equations. Variational and energy methods provide an alternate approach to the solution of this problem. Prior to the study of this methods, some of the basic mathematical concepts used in variational methods will be presented.

7.1.1 Stationary point of a function

Consider a funtion of n variables, $F = F(u_1, u_2, \dots, u_n)$. The stationary points [3] of this function are defined as those for which

$$\frac{\partial F}{\partial u_i} = 0, \quad i = 1, 2, \dots, n. \quad (7.1)$$

For a function of a single variable, this condition corresponds to a horizontal tangent to the curve, as illustrated in fig. 7.1. At a stationary point, the function can present a minimum, a maximum, or a saddle point.

If a function is stationary at a point, conditions (7.1) hold, and the following statement

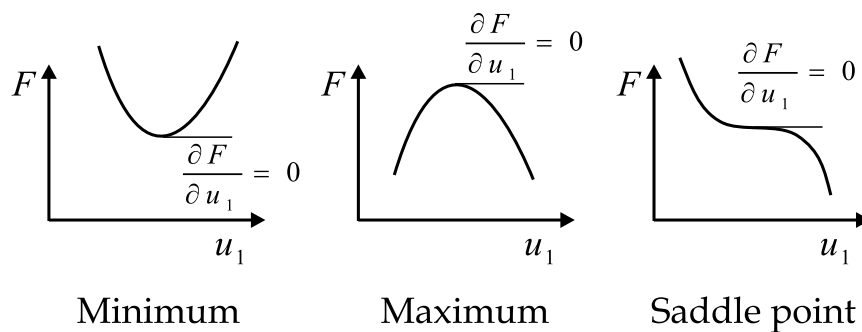


Figure 7.1: Stationary points of a function.

is then true

$$\frac{\partial F}{\partial u_1} w_1 + \frac{\partial F}{\partial u_2} w_2 + \dots + \frac{\partial F}{\partial u_n} w_n = 0, \quad (7.2)$$

where $w_1, w_2, \dots, w_n$ are arbitrary quantities. It is convenient to use a special notation for these arbitrary quantities, $w_i = \delta u_i$, where δu_i is called a *virtual change* in u_i . The above statement becomes

$$\frac{\partial F}{\partial u_1} \delta u_1 + \frac{\partial F}{\partial u_2} \delta u_2 + \dots + \frac{\partial F}{\partial u_n} \delta u_n = 0. \quad (7.3)$$

The “variation in F ”, noted δF , is defined as

$$\delta F = \frac{\partial F}{\partial u_1} \delta u_1 + \frac{\partial F}{\partial u_2} \delta u_2 + \dots + \frac{\partial F}{\partial u_n} \delta u_n. \quad (7.4)$$

It follows that

$$\delta F = 0 \quad (7.5)$$

at a stationary point. The differential condition, eq. (7.1), and the variational condition, eq. (7.5) must both hold at a stationary point. From the above developments, it is clear that eq. (7.1) implies eq. (7.5) and since the above reasoning can be reversed, it is simple to prove that eq. (7.5) implies eq. (7.1). Hence, the two conditions are entirely equivalent.

To determine whether a stationary point is a minimum, a maximum, or a saddle point it is necessary to consider the second derivatives [3] of the functions. If

$$\sum_{i,j=1,n} \frac{\partial^2 F}{\partial u_i \partial u_j} du_i du_j > 0 \quad (7.6)$$

at a stationary point for all increments du_i and du_j , the function presents a minimum. If, on the other hand, the same quantity is negative for all du_i and du_j , the function presents a maximum. Finally, if the same quantity can be positive or negative depending on the choice of the increments, the function presents a saddle point. From the definition of δF , eq. (7.4), it follows that

$$\delta^2 F = \sum_{i,j=1,n} \frac{\partial^2 F}{\partial u_i \partial u_j} \delta u_i \delta u_j. \quad (7.7)$$

It is now clear that a stationary point is a minimum if

$$\delta^2 F > 0 \quad (7.8)$$

for all arbitrary variations δu_i and δu_j . It is a maximum if $\delta^2 F < 0$ for all variations, and a saddle point occurs if the sign of the second variation depends on the choice of the variations.

7.1.2 Lagrange multiplier method

Consider once more the problem of determining a stationary point of a function of several variables $F = F(u_1, u_2, \dots, u_n)$. However, in this case, the variables are not independent, rather they are subjected to a constraint of the form

$$f(u_1, u_2, \dots, u_n) = 0. \quad (7.9)$$

Conceptually one could use the constraint to express one variable, say u_n , in term of the others. Then, u_n is eliminated from F to obtain a function of $n - 1$ independent variables $F = F(u_1, u_2, \dots, u_{n-1})$, a problem identical to that treated in the above section. In many practical situations it might be cumbersome, or even impossible, to completely eliminate one variable of the problem.

This elimination process can be avoided by using an alternate approach. At a stationary point, the variation of F vanishes

$$\delta F = \frac{\partial F}{\partial u_1} \delta u_1 + \frac{\partial F}{\partial u_2} \delta u_2 + \dots + \frac{\partial F}{\partial u_n} \delta u_n = 0. \quad (7.10)$$

However this statement does not imply $\partial F / \partial u_i = 0$ for $i = 1, 2, \dots, n$ because the variations δu_i cannot be chosen arbitrarily since they must satisfy the constraint, eq. (7.9). The relation among the variations δu_i can be explicitly written by taking a variation of the constraint to find

$$\delta f = \frac{\partial f}{\partial u_1} \delta u_1 + \frac{\partial f}{\partial u_2} \delta u_2 + \dots + \frac{\partial f}{\partial u_n} \delta u_n = 0. \quad (7.11)$$

A linear combination of eqs. (7.10) and (7.11) yields

$$\frac{\partial F}{\partial u_1} \delta u_1 + \dots + \frac{\partial F}{\partial u_n} \delta u_n + \lambda \left[\frac{\partial f}{\partial u_1} \delta u_1 + \dots + \frac{\partial f}{\partial u_n} \delta u_n \right] = 0, \quad (7.12)$$

where λ is an arbitrary function of the variables $u_1, u_2, \dots, u_n$ called *Lagrange multiplier*. Regrouping the various terms then leads to

$$\sum_{i=1}^n \left[\frac{\partial F}{\partial u_i} + \lambda \frac{\partial f}{\partial u_i} \right] \delta u_i = 0. \quad (7.13)$$

Conceptually, one could now express δu_n in term of the $n - 1$ other variations δu_i to be left with $n - 1$ independent, arbitrary variations. To avoid this cumbersome step, the arbitrary Lagrange multiplier is chosen such that

$$\frac{\partial F}{\partial u_n} + \lambda \frac{\partial f}{\partial u_n} = 0. \quad (7.14)$$

With this choice, the last term of the sum in eq. 7.13 vanishes for all δu_n . Hence, there is no need to express this variation in terms of the $n - 1$ others which can now be treated as independent, arbitrary quantities, implying

$$\frac{\partial F}{\partial u_i} + \lambda \frac{\partial f}{\partial u_i} = 0, \quad i = 1, 2, \dots, n - 1. \quad (7.15)$$

These last two equations imply that

$$\delta F + \lambda \delta f = 0, \quad (7.16)$$

where the n variables u_i are considered independent. Finally,

$$\delta F + \lambda \delta f = \delta F + \lambda \delta f + f \delta \lambda = \delta(F + \lambda f) = 0, \quad (7.17)$$

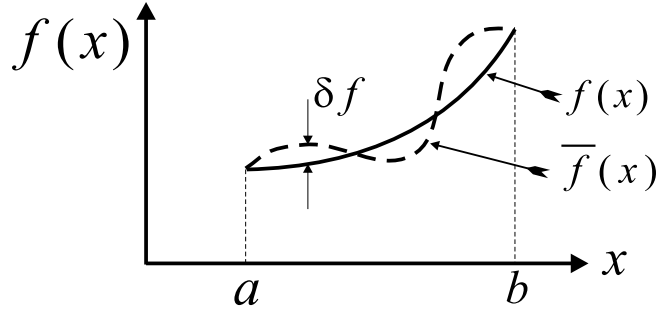


Figure 7.2: The concept of variation of a function.

since $f\delta\lambda = 0$ for arbitrary $\delta\lambda$ in view of the constraint, eq. (7.9).

In summary, the initial constrained problem can be replaced by an *unconstrained problem*

$$\delta F^* = 0, \quad F^* = F + \lambda f. \quad (7.18)$$

It is important to note that the initial problem involves n variables u_i and one constraint. The final problem involves $n + 1$ variables, u_i for $i = 1, 2, \dots, n$ and the Lagrange multiplier λ , but no constraint.

7.1.3 Stationary point of a definite integral

Next, the determination of the stationary point of the following definite integral

$$I = \int_a^b F(y, y', x) dx \quad (7.19)$$

is considered, where the notation $()'$ is used to indicate a derivatives with respect to x . y is an unknown function subjected to boundary conditions $y(a) = \alpha$, $y(b) = \beta$. At first, this problem seems to be of a completely different nature from those treated in the previous sections. Indeed, I is a “function of a function”, *i.e.* the value of the definite integral I depends on the choice of the unknown function y . Since there are an infinite number of values of y between a and b , I is equivalent to a function of an infinite number of variables.

This problem will be treated using the variational formalism introduced in the previous sections. First, the concept of *variation of a function*, denoted δf is introduced. Fig. 7.2 shows two functions $f(x)$ and $\bar{f}(x)$ such that

$$\delta f = \bar{f}(x) - f(x) = \phi(x), \quad (7.20)$$

where $\phi(x)$ is a continuous and differentiable, but otherwise arbitrary function such that $\phi(a) = \phi(b) = 0$. In other words, δf is a virtual change that brings the function $f(x)$ to a new, arbitrary function $\bar{f}(x)$. Note that $\delta f(a) = \delta f(b) = 0$.

The stationarity of I requires

$$\delta I = \delta \int_a^b F(y, y', x) dx = \int_a^b \delta F(y, y', x) dx = 0. \quad (7.21)$$

With the help of eq. (7.4), this becomes

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right] dx = 0. \quad (7.22)$$

Integration by parts is now applied to the second term in the square bracket

$$\int_a^b \frac{\partial F}{\partial y'} \delta \left(\frac{dy}{dx} \right) dx = \int_a^b \frac{\partial F}{\partial y'} \frac{d}{dx} (\delta y) dx = - \int_a^b \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \delta y dx + \left[\frac{\partial F}{\partial y'} \delta y \right]_a^b. \quad (7.23)$$

The boundary term vanishes since $\delta y(a) = \delta y(b) = 0$, and the stationarity condition then becomes

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] \delta y dx = 0. \quad (7.24)$$

The bracketed term must vanish because the integral must go to zero for *all arbitrary variations* δy . This yields

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0. \quad (7.25)$$

Here again, the above reasoning can be reversed. Starting from eq. (7.25), and performing the integration by parts in the reversed order implies $\delta I = 0$. In summary, *the necessary and sufficient condition for the definite integral to be at a stationary point is that eq. (7.25) be satisfied*. These differential equations are called the *Euler-Lagrange equations* of the problem.

The variational formalism introduced in this section will be systematically applied to elasticity problem. It will be shown that the equations of elasticity can be viewed as the Euler-Lagrange equations associated with the stationarity condition of definite integrals. Various forms of the equations of elasticity can be easily obtained by direct manipulations of these definite integrals. It is therefore important to understand the variational formalism and its implications.

A crucial difference exists between an increment df of a function $f(x)$ and a variation δf of the same function, as depicted in fig. 7.3. df is an infinitesimal change in $f(x)$ resulting from an infinitesimal change dx in the independent variable; df/dx represents the tangent at the point. On the other hand, δf is an arbitrary virtual change that brings $f(x)$ to $\bar{f}(x)$. The two quantities, df and δf , are clearly unrelated, the former is positive in fig. 7.3, whereas the latter is negative.

Although the concepts associated with the notation df and δf are clearly distinct, the manipulations of the two symbols are quite similar. For instance, the order of application of the two operations can be interchanged. Indeed,

$$\frac{d}{dx}(\delta f) = \frac{d}{dx}(\bar{f} - f) = \frac{d\bar{f}}{dx} - \frac{df}{dx} = \delta \left(\frac{df}{dx} \right). \quad (7.26)$$

Similarly, the order of the integration and variation operations commute

$$\delta \int_a^b F dx = \int_a^b \bar{F} dx - \int_a^b F dx = \int_a^b (\bar{F} - F) dx = \int_a^b \delta F dx. \quad (7.27)$$

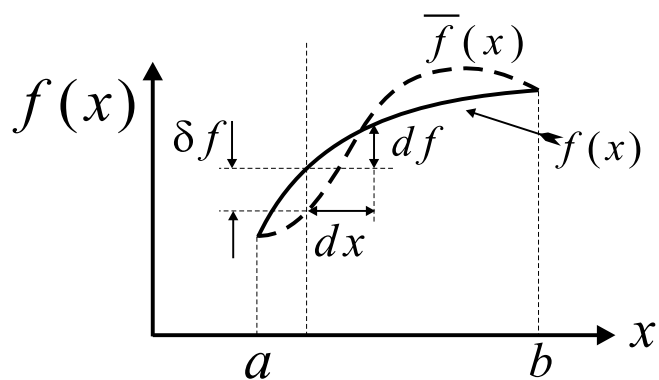


Figure 7.3: The difference between an increment df and a variation δf .

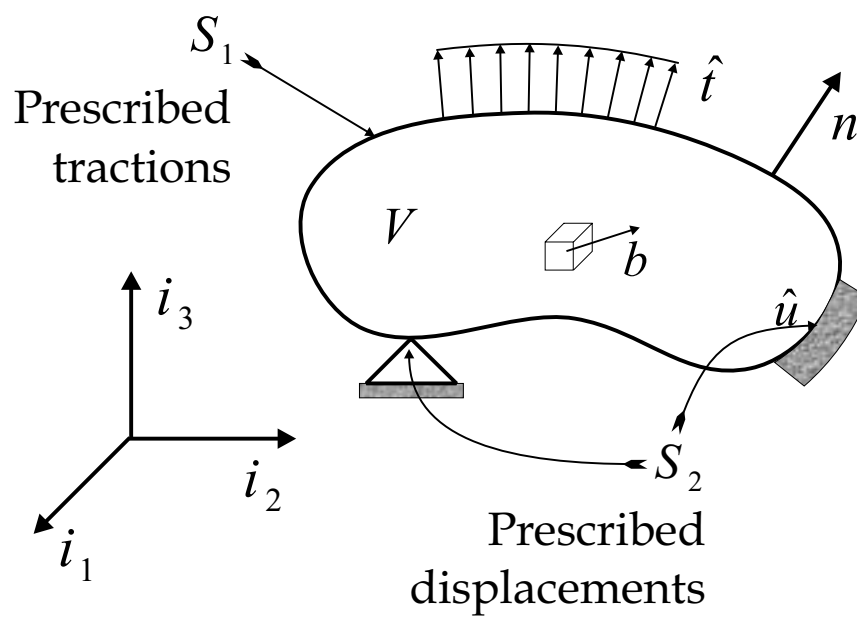


Figure 7.4: General elasticity problem.

7.2 Variational and Energy Principles

Consider a general elasticity problem consisting of an elastic body of arbitrary shape subjected to surface tractions and body forces as well as geometric boundary conditions such as prescribed displacements at a point or over a portion of its outer surface, as depicted in fig. 7.4 . The volume of the body is denoted $\mathcal{V}$ and its outer surface $\mathcal{S}$. The outer normal to $\mathcal{S}$ is the unit vector $\mathbf{n}$. $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the portions of the outer surface where prescribed tractions $\hat{\mathbf{t}}$ and prescribed displacements $\hat{\mathbf{u}}$ are applied, respectively. At a point of the outer surface, either tractions or displacements can be prescribed, but it is impossible to prescribe both together. Consequently, $\mathcal{S}_1$ and $\mathcal{S}_2$ share no common points and $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$. Note that a point of the outer surface that is traction free belongs to $\mathcal{S}_1$, since zero tractions $\hat{\mathbf{t}} = 0$ are prescribed at that point. Body forces $\mathbf{b}$ might also be applied over the entire volume of the body. Gravity forces are a typical example of body forces, but such forces can also arise as a result of electric or magnetic fields. In dynamic problems, inertial forces will also be applied as body forces in accordance with D'Alembert's principle.

The basic equations of elasticity developed in chapter 1 form a set of partial differential equations that can be solved to find the displacement, strain, and stress fields at all points in $\mathcal{V}$. These equations will be reviewed in section 7.2.1 where several important definitions are also introduced. In the subsequent sections, a number of variational and energy principles are presented that provide an alternate formalism for the solution of elasticity problems. This formalism is the basis for powerful numerical techniques, such as the *finite element method*, that are routinely used to obtain approximate solutions to complex elasticity problems.

7.2.1 Review of the equations of linear elasticity

The equations of elasticity can be broken into three groups

1. The *equations of equilibrium* are the most fundamental equations. They were derived in sections 1.1.2 and 1.1.3 from Newton's law stating that the sum of all the forces acting on a differential element of the structure should vanish.
2. The *strain-displacement equations* merely define the strain components that are used for the characterization of the deformation of the body at a point. The strain-displacement relationships were derived in section 1.3.1 from purely geometric considerations.
3. The *constitutive laws* relate the stress and strain components. They consist of a mathematical idealization of the experimentally observed behavior of materials. The homogeneous, isotropic, linear elastic material behavior described in section 1.5.1 is a commonly used, but highly idealized constitutive law. Many materials can present one or more of the following features: anisotropy, plasticity, viscoelasticity, or creep, to name just a few commonly observed material behaviors.

The equilibrium equations for a differential element of the body are

$$\begin{aligned}\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 &= 0; \\ \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 &= 0; \\ \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 &= 0,\end{aligned}\tag{7.28}$$

and must be satisfied at all points in $\mathcal{V}$. The surface equilibrium equations are

$$t_1 = \hat{t}_1; \quad t_2 = \hat{t}_2; \quad t_3 = \hat{t}_3,\tag{7.29}$$

where the surface tractions were defined in eq. (1.14). The surface equilibrium equations are also called the *force*, or *natural boundary conditions*. The stress array

$$\underline{\sigma}^T = [\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{13}, \tau_{12}],\tag{7.30}$$

will be used whenever that notation is convenient.

Definition 1 A stress field $\underline{\sigma}(x_1, x_2, x_3)$ is said to be **statically admissible** if it satisfies the equilibrium equations, eqs. (7.28), at all points in $\mathcal{V}$ and the surface equilibrium equations, eqs. (7.29), at all points on $\mathcal{S}_1$.

When the displacements are small, it is convenient to use the engineering strain components to measure the deformation at a point. The axial and shearing strain components are related to the displacements as

$$\begin{aligned}\varepsilon_1 &= \frac{\partial u_1}{\partial x_1}; \quad \varepsilon_2 = \frac{\partial u_2}{\partial x_2}; \quad \varepsilon_3 = \frac{\partial u_3}{\partial x_3}, \\ \gamma_{23} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}; \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}; \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1},\end{aligned}\tag{7.31}$$

respectively. In order to compute strain components, the displacements field must be continuous and differentiable. Furthermore, the displacements must be equal to the prescribed displacements over $\mathcal{S}_2$

$$u_1 = \hat{u}_1; \quad u_2 = \hat{u}_2; \quad u_3 = \hat{u}_3;\tag{7.32}$$

these are called the *geometric boundary conditions*. The strain array

$$\underline{\varepsilon}^T = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{23}, \gamma_{13}, \gamma_{12}],\tag{7.33}$$

will be used whenever that notation is convenient.

Definition 2 A displacement field $\mathbf{u}(x_1, x_2, x_3)$ is said to be **kinematically admissible** if it is continuous and differentiable at all points in $\mathcal{V}$ and satisfies the geometric boundary conditions, eqs. (7.32), at all points on $\mathcal{S}_2$.

Definition 3 A strain field $\underline{\varepsilon}(x_1, x_2, x_3)$ is said to be **compatible** if it is derived from a kinematically admissible displacement field through the strain-displacement relationships, eqs. (7.31).

Finally, the stress and strain fields must satisfy the constitutive laws at all points in $\mathcal{V}$. For some materials, a simple linear relationship can be assumed to hold when the strain components remain small; this writes

$$\underline{\sigma} = C \underline{\varepsilon}, \text{ or } \underline{\varepsilon} = S \underline{\sigma}, \quad (7.34)$$

where C is a symmetric, positive-definite stiffness matrix, and S a symmetric, positive-definite compliance matrix.

In summary, the complete solution of an elasticity problem involves

1. a statically admissible stress field,
2. a kinematically admissible displacement field and the corresponding compatible strain field,
3. and a constitutive law satisfied at all points in $\mathcal{V}$.

7.2.2 The principle of virtual work

Consider an elastic body that is in equilibrium under applied body forces and surface tractions. This implies that the stress field is statically admissible, *i.e.* the equilibrium equations, eqs. (7.28), are satisfied at all points in $\mathcal{V}$ and the surface equilibrium equations, eqs. (7.29), at all points on $\mathcal{S}_1$. The following statement is now constructed

$$\begin{aligned} - \int_{\mathcal{V}} \left[\left(\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + b_1 \right) \delta u_1 + \left(\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + b_2 \right) \delta u_2 \right. \\ \left. + \left(\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + b_3 \right) \delta u_3 \right] d\mathcal{V} + \int_{\mathcal{S}_1} (\mathbf{t} - \hat{\mathbf{t}}) \cdot \delta \mathbf{u} d\mathcal{S} = 0. \end{aligned} \quad (7.35)$$

This statement was constructed in the following manner. Each of the three equilibrium equations was multiplied by an arbitrary, virtual change in displacement, then integrated over the range of validity of the equation, $\mathcal{V}$. Similarly, each of the three surface equilibrium equations was multiplied by an arbitrary, virtual change in displacement, then integrated over the range of validity of the equation, $\mathcal{S}_1$. Since the stress field is statically admissible, each term in parenthesis is zero, and multiplication by an arbitrary quantity results in a zero product. Each of the two integral then vanishes, as does their sum. The dot product notation $(\mathbf{t} - \hat{\mathbf{t}}) \cdot \delta \mathbf{u} = (t_1 - \hat{t}_1)\delta u_1 + (t_2 - \hat{t}_2)\delta u_2 + (t_3 - \hat{t}_3)\delta u_3$ was introduced to obtain compact expressions.

Next, integration by parts is performed. For the first term of the volume integral this operation is

$$- \int_{\mathcal{V}} \frac{\partial \sigma_1}{\partial x_1} \delta u_1 d\mathcal{V} = \int_{\mathcal{V}} \sigma_1 \frac{\partial \delta u_1}{\partial x_1} d\mathcal{V} - \int_{\mathcal{S}} n_1 \sigma_1 \delta u_1 d\mathcal{S}, \quad (7.36)$$

where n_1 is the component of the outward unit normal along $\mathbf{i}_1$, see fig. 7.4. A similar operation is performed on each stress derivative terms appearing in eq. (7.35) to yield

$$\int_{\mathcal{V}} \underline{\sigma}^T \delta \underline{\varepsilon} \, d\mathcal{V} - \int_{\mathcal{V}} \mathbf{b} \cdot \delta \mathbf{u} \, d\mathcal{V} - \int_{\mathcal{S}} \mathbf{t} \cdot \delta \mathbf{u} \, d\mathcal{S} + \int_{\mathcal{S}_1} (\mathbf{t} - \hat{\mathbf{t}}) \cdot \delta \mathbf{u} \, d\mathcal{S} = 0, \quad (7.37)$$

where the *virtual, compatible strain field* was defined as

$$\begin{aligned} \delta \varepsilon_1 &= \frac{\partial \delta u_1}{\partial x_1}; & \delta \varepsilon_2 &= \frac{\partial \delta u_2}{\partial x_2}; & \delta \varepsilon_3 &= \frac{\partial \delta u_3}{\partial x_3}, \\ \delta \gamma_{23} &= \frac{\partial \delta u_2}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_2}; & \delta \gamma_{13} &= \frac{\partial \delta u_1}{\partial x_3} + \frac{\partial \delta u_3}{\partial x_1}; & \delta \gamma_{12} &= \frac{\partial \delta u_1}{\partial x_2} + \frac{\partial \delta u_2}{\partial x_1}. \end{aligned} \quad (7.38)$$

The virtual displacements are now chosen to be kinematically admissible, which implies $\delta \mathbf{u} = 0$ on $\mathcal{S}_2$. Expression (7.37) now simplifies to

$$\int_{\mathcal{V}} \underline{\sigma}^T \delta \underline{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} \mathbf{b} \cdot \delta \mathbf{u} \, d\mathcal{V} + \int_{\mathcal{S}_1} \hat{\mathbf{t}} \cdot \delta \mathbf{u} \, d\mathcal{S}. \quad (7.39)$$

The term on the left hand side of this expression can be interpreted as the virtual work done by the internal stresses whereas that on the right corresponds to the work done by the externally applied loads. Eq. (7.39) can thus be interpreted as follows

Principle 1 (Principle of Virtual Work) *A body is in equilibrium if the virtual work done by the internal stresses equals the virtual work done by the externally applied loads, for all arbitrary kinematically admissible virtual displacements fields and corresponding compatible strain field.*

It has been shown thus far that if a stress field is statically admissible, the principle of virtual work must hold. It can also be shown that if the principle of virtual work holds true, then the stress field must be statically admissible. Indeed, this principle implies eq. (7.37), which in turns implies eq. (7.35) by reversing the integration by parts process. Finally, the volume and surface equilibrium equations are recovered because eq. (7.35) must hold for all arbitrary, kinematically admissible virtual displacements fields. In summary, the equations of equilibrium, eqs. (7.28) and (7.29), and the principle of virtual work are two entirely equivalent statements. Because the principle of virtual work is solely a statement of equilibrium, it is always true. However, for the solution of specific elasticity problems, it must be complemented with stress-strain relationships and constitutive laws.

It is interesting to compare the present statement of the principle of virtual work with that derived in chapter 3 for beams under axial and transverse loads, eqs. (3.108) and (3.109), respectively. The statements are different because the present formulation deals with general, three-dimensional stress states, whereas the formulation of chapter 3 deals with the stress resultants associated with beam theory. However, the physical interpretation of the various statement is identical in all cases.

7.2.3 The principle of complementary virtual work

Consider an elastic body undergoing kinematically admissible displacements and compatible strains. This implies that the strain-displacement relationships, eqs. (7.31), are satisfied at

all points in $\mathcal{V}$ and the geometric boundary conditions, eqs. (7.32), at all points on $\mathcal{S}_2$. The following statement is now constructed

$$\begin{aligned} & \int_{\mathcal{V}} \left[\left(\varepsilon_1 - \frac{\partial u_1}{\partial x_1} \right) \delta \sigma_1 + \left(\varepsilon_2 - \frac{\partial u_2}{\partial x_2} \right) \delta \sigma_2 + \left(\varepsilon_3 - \frac{\partial u_3}{\partial x_3} \right) \delta \sigma_3 + \left(\gamma_{23} - \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right) \delta \tau_{23} \right. \\ & \left. + \left(\gamma_{13} - \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \delta \tau_{13} + \left(\gamma_{12} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right) \delta \tau_{12} \right] d\mathcal{V} + \int_{\mathcal{S}_2} (\mathbf{u} - \hat{\mathbf{u}}) \cdot \delta \mathbf{t} d\mathcal{S} = 0. \end{aligned} \quad (7.40)$$

This statement was constructed in the following manner. Each of the six strain-displacement relationships was multiplied by an arbitrary, virtual change in stress, then integrated over the range of validity of the equations, $\mathcal{V}$. Similarly, each of the three geometric boundary conditions was multiplied by an arbitrary, virtual change in surface traction, then integrated over the range of validity of the equation, $\mathcal{S}_2$. Since the strain field is compatible and the displacement field kinematically admissible, each term in parenthesis is zero, and multiplication by an arbitrary quantity results in a zero product. Each of the two integral then vanishes, as does their sum.

Next, integration by parts is performed. For the first term of the volume integral this operation is

$$- \int_{\mathcal{V}} \frac{\partial u_1}{\partial x_1} \delta \sigma_1 d\mathcal{V} = \int_{\mathcal{V}} u_1 \frac{\partial \delta \sigma_1}{\partial x_1} d\mathcal{V} - \int_{\mathcal{S}} u_1 n_1 \delta \sigma_1 d\mathcal{S}, \quad (7.41)$$

where n_1 is the component of the outward unit normal along $\mathbf{i}_1$, see fig. 7.4. A similar operation is performed on each displacement derivative terms appearing in eq. (7.40) to yield

$$\begin{aligned} & \int_{\mathcal{V}} \underline{\varepsilon}^T \delta \underline{\sigma} d\mathcal{V} + \int_{\mathcal{V}} \left[\left(\frac{\partial \delta \sigma_1}{\partial x_1} + \frac{\partial \delta \tau_{21}}{\partial x_2} + \frac{\partial \delta \tau_{31}}{\partial x_3} \right) u_1 + \left(\frac{\partial \delta \tau_{12}}{\partial x_1} + \frac{\partial \delta \sigma_2}{\partial x_2} + \frac{\partial \delta \tau_{32}}{\partial x_3} \right) u_2 \right. \\ & \left. + \left(\frac{\partial \delta \tau_{13}}{\partial x_1} + \frac{\partial \delta \tau_{23}}{\partial x_2} + \frac{\partial \delta \sigma_3}{\partial x_3} \right) u_3 \right] d\mathcal{V} - \int_{\mathcal{S}} \mathbf{u} \cdot \delta \mathbf{t} d\mathcal{S} + \int_{\mathcal{S}_2} (\mathbf{u} - \hat{\mathbf{u}}) \cdot \delta \mathbf{t} d\mathcal{S} = 0, \end{aligned} \quad (7.42)$$

The virtual stresses are now chosen to be statically admissible, which implies that virtual stresses satisfy the equilibrium equations, and $\delta \mathbf{t} = 0$ on $\mathcal{S}_1$. Expression (7.42) now becomes

$$\int_{\mathcal{V}} \underline{\varepsilon}^T \delta \underline{\sigma} d\mathcal{V} - \int_{\mathcal{S}_2} \hat{\mathbf{u}} \cdot \delta \mathbf{t} d\mathcal{S} = 0. \quad (7.43)$$

This statement can be interpreted as follows

Principle 2 (Principle of Complementary Virtual Work) *A body is undergoing kinematically admissible displacements and compatible strains if the above quantity vanishes for all statically admissible virtual stress fields.*

It has been shown thus far that if a body undergoing kinematically admissible displacements and compatible strains, the principle of complementary virtual work must hold. It can also be shown that if the principle of complementary virtual work holds true, then the displacement field must be kinematically admissible and the strain field compatible. Indeed, this principle implies eq. (7.42), which in turns implies eq. (7.40) by reversing the integration

by parts process. Finally, the strain-displacement relationships and geometric boundary conditions are recovered because eq. (7.40) must hold for all arbitrary stress virtual stress fields. In summary, the strain-displacement relationships and the geometric boundary conditions, eqs. (7.31) and (7.32), respectively, and the principle of complementary virtual work are two entirely equivalent statements.

7.2.4 The principle of minimum total potential energy

Constitutive laws consist of a mathematical idealization of the experimentally observed behavior of materials, and can take various different forms to capture the numerous types of material behavior observed in nature. For some types of material behavior a scalar state function can be assumed to exist such that

$$\underline{\sigma} = \frac{\partial a(\underline{\varepsilon})}{\partial \underline{\varepsilon}}, \quad (7.44)$$

where $a(\underline{\varepsilon})$ is called the *strain energy density function*. For instance, the strain energy density function of a linear elastic material is

$$a(\underline{\varepsilon}) = \frac{1}{2} \underline{\varepsilon}^T C \underline{\varepsilon}. \quad (7.45)$$

Introducing this function into eq. (7.44), yields $\underline{\sigma} = C \underline{\varepsilon}$, the constitutive law for a linear elastic material.

Introducing this new form of constitutive laws into the left hand side of the principle of virtual work, eq. (7.39), leads to

$$\int_{\mathcal{V}} \delta \underline{\varepsilon}^T \underline{\sigma} \, d\mathcal{V} = \int_{\mathcal{V}} \delta \underline{\varepsilon}^T \frac{\partial a(\underline{\varepsilon})}{\partial \underline{\varepsilon}} \, d\mathcal{V} = \int_{\mathcal{V}} \delta a(\mathbf{u}) \, d\mathcal{V} = \delta \int_{\mathcal{V}} a(\mathbf{u}) \, d\mathcal{V} = \delta A(\mathbf{u}), \quad (7.46)$$

where the chain rule for derivatives was used. The strain energy density and the *total strain energy* of the body, $A = \int_{\mathcal{V}} a \, d\mathcal{V}$, must be expressed in terms of the displacement field $\mathbf{u}$ using the strain displacement relationships because the principle of virtual work requires a compatible strain field. The principle of virtual work now writes

$$\delta A(\mathbf{u}) = \int_{\mathcal{V}} \mathbf{b} \cdot \delta \mathbf{u} \, d\mathcal{V} + \int_{S_1} \hat{\mathbf{t}} \cdot \delta \mathbf{u} \, dS. \quad (7.47)$$

Next, the body forces and surface tractions are assumed to be derivable from potential functions

$$\mathbf{b} = -\frac{\partial \phi}{\partial \mathbf{u}}; \quad \hat{\mathbf{t}} = -\frac{\partial \psi}{\partial \mathbf{u}}, \quad (7.48)$$

where ϕ is the *potential of the body forces*, and ψ the *potential of the surface tractions*. For instance, the potential of fixed surface tractions is simply $\psi = -\hat{\mathbf{t}} \cdot \mathbf{u}$, whereas the potential of the body forces associated with a gravity field $\mathbf{g}$ is $\phi = -m \mathbf{g} \cdot \mathbf{u}$, where m is the mass per unit volume of the body. It is important to note that potential functions do not exist for all types of forces. Potential functions do not always exist for forces that depend on the orientation of the body, such as follower forces or aerodynamic forces, for instance.

The right hand side of the principle of virtual work, eq. (7.39), now becomes

$$\begin{aligned} \int_{\mathcal{V}} \mathbf{b} \cdot \delta \mathbf{u} \, d\mathcal{V} + \int_{S_1} \hat{\mathbf{t}} \cdot \delta \mathbf{u} \, d\mathcal{S} &= - \int_{\mathcal{V}} \frac{\partial \phi}{\partial \mathbf{u}} \cdot \delta \mathbf{u} \, d\mathcal{V} - \int_{S_1} \frac{\partial \psi}{\partial \mathbf{u}} \cdot \delta \mathbf{u} \, d\mathcal{S} \\ &= - \int_{\mathcal{V}} \delta \phi(\mathbf{u}) \, d\mathcal{V} - \int_{S_1} \delta \psi(\mathbf{u}) \, d\mathcal{S} = -\delta \int_{\mathcal{V}} \phi(\mathbf{u}) \, d\mathcal{V} - \delta \int_{S_1} \psi(\mathbf{u}) \, d\mathcal{S} = -\delta \Phi(\mathbf{u}), \end{aligned} \quad (7.49)$$

where $\Phi(\mathbf{u}) = \int_{\mathcal{V}} \phi(\mathbf{u}) \, d\mathcal{V} + \int_{S_1} \psi(\mathbf{u}) \, d\mathcal{S}$ is the total potential the applied loads. Introducing this result into eq. (7.47) leads to

$$\delta A(\mathbf{u}) = -\delta \Phi(\mathbf{u}), \text{ or } \delta (A(\mathbf{u}) + \Phi(\mathbf{u})) = 0. \quad (7.50)$$

The *total potential energy* of the body is now defined as

$$\Pi(\mathbf{u}) = A(\mathbf{u}) + \Phi(\mathbf{u}), \quad (7.51)$$

and it follows that

$$\delta \Pi(\mathbf{u}) = 0. \quad (7.52)$$

This statement can be interpreted as follows

Principle 3 (Principle of Minimum Total Potential Energy) *Among all kinematically admissible displacements fields, the actual displacement field that corresponds to the equilibrium configuration of the body makes Π an absolute minimum.*

Eq. (7.52) only proves that at equilibrium, the total potential energy presents a stationary point. As discussed in section 7.1.1, the sign of the second variation $\delta^2 \Pi$ will determine whether the stationary point actually is a minimum. The first variation in Π is

$$\delta \Pi(\mathbf{u}) = \int_{\mathcal{V}} \sum_{i=1}^6 \frac{\partial a}{\partial \varepsilon_i} \delta \varepsilon_i \, d\mathcal{V} - \int_{\mathcal{V}} \mathbf{b} \cdot \delta \mathbf{u} \, d\mathcal{V} - \int_{S_1} \hat{\mathbf{t}} \cdot \delta \mathbf{u} \, d\mathcal{S}, \quad (7.53)$$

and its second variation is then

$$\delta^2 \Pi(\mathbf{u}) = \int_{\mathcal{V}} \sum_{i,j=1}^6 \frac{\partial^2 a}{\partial \varepsilon_i \partial \varepsilon_j} \delta \varepsilon_i \delta \varepsilon_j \, d\mathcal{V}. \quad (7.54)$$

Based on physical grounds, the strain energy density function must be a positive-definite function of the strain components, which implies $\sum_{i,j=1}^6 \frac{\partial^2 a}{\partial \varepsilon_i \partial \varepsilon_j} \delta \varepsilon_i \delta \varepsilon_j \geq 0$ for all $\delta \varepsilon_i$. Indeed, if the strain energy function were not positive-definite, strain states would exist that generate a negative strain energy, *i.e.* the elastic body would generate energy under deformation, a situation that is physically impossible. It follows that $\delta^2 \Pi \geq 0$, and hence, Π present an absolute minimum at its stationary point.

If the principle of minimum total potential energy holds true, the total potential energy must present a stationary point, implying eq. (7.50). In turns, this equation implies the principle of virtual work, in which the stresses were expressed in terms of the strains using constitutive laws of the form (7.44), and strains were themselves expressed in terms of displacements using the strain-displacement relationships. In summary, the principle of

minimum total potential energy implies the equations of equilibrium of the problem expressed in terms of the sole displacement field. Such equations are the Euler-Lagrange equations corresponding to the stationarity condition for the total potential energy, see section 7.1.3. The principle of minimum total potential energy implies the principle of virtual work, but the principle of virtual work only implies the principle of minimum total potential energy under restrictive assumptions on existence of a strain energy density function and of potentials of the body forces and surface tractions.

7.2.5 The strain energy

The principle of minimum total potential energy establishes the total strain energy of a body as quantity of great importance. The total strain energy for beam was already evaluated for beams subjected axial strains, eq.(3.33), to curvature, eq. (3.66), or twist rate, eq. (5.35).

Consider a differential element of a linear elastic solid subjected to a single axial stress component, say σ_1 . Under the effect of these stresses, the two faces on which they act undergo a relative displacement du_1 . Since this deformation is proportional to the applied stress, the work dW done by the axial force $\sigma_1 dx_2 dx_3$ as it increases from zero to its present value is

$$dW = \frac{1}{2} (\sigma_1 dx_2 dx_3) du_1 = \frac{1}{2} \sigma_1 \frac{\partial u_1}{\partial x_1} dx_1 dx_2 dx_3 = \frac{1}{2} \sigma_1 \varepsilon_1 dx_1 dx_2 dx_3.$$

This work is given by the area under the stress-strain curve for the material, a straight line for the assumed linear elastic material behavior. If the material can be assumed to follow Hooke's law, eq. (1.82), this work becomes

$$dW = \frac{1}{2} E \varepsilon_1^2 dx_1 dx_2 dx_3; \quad (7.55)$$

in this case the strain energy density function is $a(\varepsilon_1) = 1/2 E \varepsilon_1^2$.

Consider now the more general case where the same differential element is subjected to a complete three-dimensional stress state. Since the various stress components act in mutually orthogonal directions, the work they each perform can be added to yield the total work as

$$dW = \frac{1}{2} [\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_3 \varepsilon_3 + \tau_{23} \gamma_{23} + \tau_{13} \gamma_{13} + \tau_{12} \gamma_{12}] dx_1 dx_2 dx_3 = \frac{1}{2} \underline{\sigma}^T \underline{\varepsilon} dx_1 dx_2 dx_3.$$

Generalized Hooke's law, eq. (1.85) is now introduced to find

$$dW = \frac{1}{2} \underline{\varepsilon}^T C \underline{\varepsilon} dx_1 dx_2 dx_3, \quad (7.56)$$

where C is the stiffness matrix for the material, given as

$$C = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}. \quad (7.57)$$

The strain energy density function for a linear elastic material that follows Hooke's law, is now

$$a(\underline{\varepsilon}) = \frac{1}{2} \underline{\varepsilon}^T C \underline{\varepsilon}. \quad (7.58)$$

In the case of plane stress state, the sole in-plane strain and stress components contribute to the strain energy expression

$$a(\underline{\varepsilon}) = \frac{1}{2} \underline{\varepsilon}^T C \underline{\varepsilon}, \quad (7.59)$$

where $\underline{\varepsilon}^T = [\varepsilon_1, \varepsilon_2, \gamma_{12}]$ are the relevant strain components. The associated 3×3 stiffness matrix is

$$C_{\text{plane stress}} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \quad (7.60)$$

Note that the strain component ε_3 does not vanish due to Poisson's ratio effects, but it does not contribute to the strain energy since the corresponding stress components σ_3 does vanish. For the plane strain case, the stiffness matrix is matrix

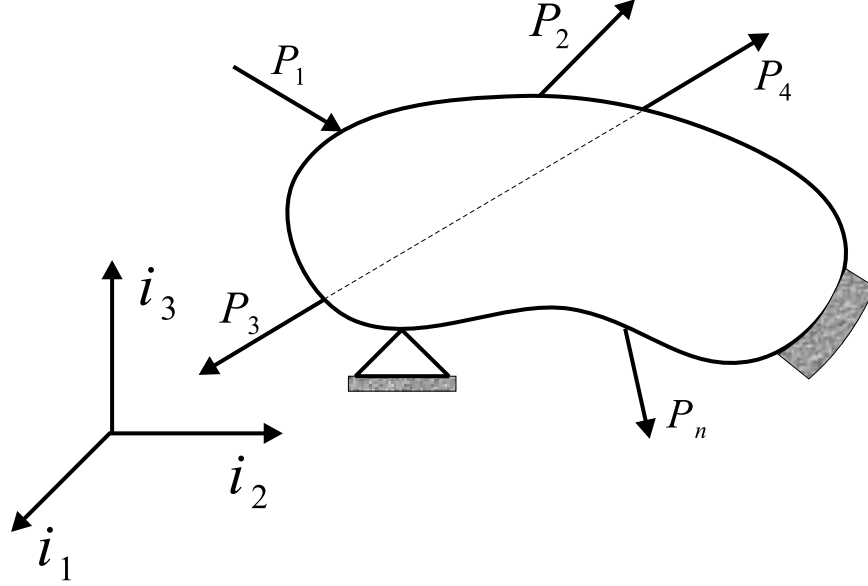
$$C_{\text{plane strain}} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}. \quad (7.61)$$

In this case, the stress component σ_3 does not vanish, but it does not contribute to the strain energy because ε_3 is null in plane strain conditions.

7.2.6 Clapeyron's theorem

Consider a linear elastic structure subjected to concentrated loads P_i , $i = 1, 2, \dots, n$, as depicted in fig. 7.5. The total potential energy of the system is

$$\Pi = A - \sum_{i=1}^n P_i u_i, \quad (7.62)$$

Figure 7.5: Elastic body subjected to concentrated loads P_i .

where u_i are the displacements of the points of application of the loads P_i projected in the direction of these loads and $A = 1/2 \int_V \underline{\varepsilon}^T C \underline{\varepsilon} dV$. The stationarity of the total potential energy then implies

$$\delta\Pi = \delta A - \sum_{i=1}^n P_i \delta u_i = \underline{\varepsilon}^T C \delta \underline{\varepsilon} - \sum_{i=1}^n P_i \delta u_i = 0. \quad (7.63)$$

Since this relationship must hold for all arbitrary virtual displacement fields and associated compatible strain fields, a valid choice is $\delta u_i = u_i$ and $\delta \underline{\varepsilon} = \underline{\varepsilon}$, where u_i and $\underline{\varepsilon}$ correspond to the equilibrium configuration of the structure. It follows that $\underline{\varepsilon}^T C \underline{\varepsilon} = \sum_{i=1}^n P_i u_i$ and

$$A = \sum_{i=1}^n \frac{P_i u_i}{2}. \quad (7.64)$$

Note that the order in which the forces are applied is immaterial, the total strain energy only depends of the magnitude of the forces and the resulting projected displacements. This statement can be interpreted as follows

Theorem 1 (Clapeyron's theorem) *The total strain energy stored in a linear elastic structure equals the sum of the half product of the applied loads by the displacements of their respective points of applications projected along these loads.*

Another interpretation of this result is: for a linear elastic structure, the work done by the externally applied forces, $\sum_{i=1}^n P_i u_i / 2$, equals the total strain energy stored in the structure.

Let the structure be subjected to couples M_i , $i = 1, 2, \dots, n$. The total potential energy of the system becomes $\Pi = A - \sum_{i=1}^n M_i \phi_i$, where ϕ_i are the rotations at the point of

application of the couples projected along the axis of the couple. For such loading, it is easily shown that

$$A = \sum_{i=1}^n \frac{M_i \phi_i}{2}. \quad (7.65)$$

If a single load P acts on the structure, Clapeyron's theorem yields the corresponding projected displacement u as

$$u = \frac{2A}{P}. \quad (7.66)$$

Finally, consider the case of a structure loaded by two forces applied along the same line action, as for instance, the forces P_3 and P_4 in fig. 7.5. The total strain energy stored in the structure is $A = (P_3 u_3 + P_4 u_4)/2$. Next these two forces are assumed to act in opposite directions with a common intensity, $P_3 = P_4 = P$. It follows that $A = P(u_3 + u_4)/2$ where $u = u_3 + u_4$ represents the relative distance between the points of application of the two forces, a positive quantity if the forces pull away from each other, and is negative in the opposite case. The total strain energy now writes

$$A = \frac{Pu}{2}. \quad (7.67)$$

In summary, each term in the statement of Clapeyron's theorem can be

1. $P_i u_i/2$: where P_i is a concentrated load and u_i the projected displacement of its point of application.
2. $M_i \phi_i/2$: where M_i is a couple and ϕ_i the projected rotation at its point of application.
3. $P_i u_i/2$: where P_i is the common value of two opposite forces sharing a common line of action and u_i the projected relative displacement of their points of application.
4. $M_i \phi_i/2$: where M_i is the common value of two opposite couples sharing a common line of action and ϕ_i the projected relative rotation at their points of application.

7.2.7 The principle of minimum complementary energy

In the previous section, the constitutive laws were expressed in terms of a strain energy density function, eq. (7.44). If the deformation involves a single strain component ε , this equation can be integrated to yield $a = \int_0^\varepsilon \sigma \, d\varepsilon$, *i.e.* the strain energy density function can be interpreted as the area under the stress-strain curve, as depicted in fig. 7.6. The area between the stress-strain curve and the vertical axis is $b = \int_0^\sigma \varepsilon \, d\sigma$ and is called the *stress energy density function*, or *complementary energy function*, since $a + b = \varepsilon \sigma$. Clearly, the existence of one energy function implies the existence of the other. The constitutive laws are readily expressed in terms of the stress energy density function

$$\underline{\varepsilon} = \frac{\partial b(\underline{\sigma})}{\partial \underline{\sigma}}. \quad (7.68)$$

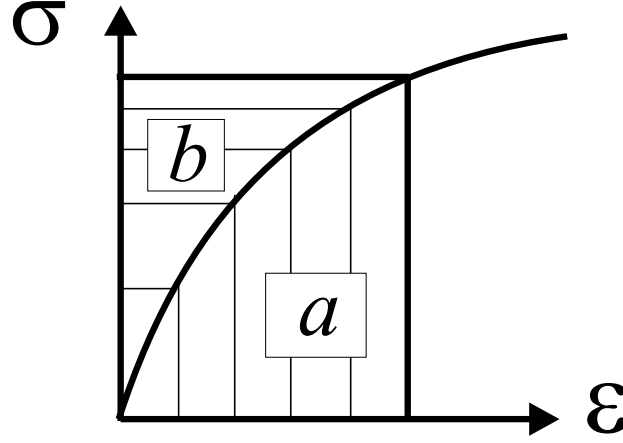


Figure 7.6: Relationship between the strain and stress energy density functions.

For instance, the stress energy density function of a linear elastic material is

$$b(\underline{\sigma}) = \frac{1}{2} \underline{\sigma}^T S \underline{\sigma}, \quad (7.69)$$

and $\underline{\varepsilon} = S \underline{\sigma}$, where S is the compliance matrix.

Introducing this new form of constitutive laws into the left hand side of the principle of complementary virtual work, eq. (7.43), leads to

$$\int_{\mathcal{V}} \delta \underline{\sigma}^T \underline{\varepsilon} \, d\mathcal{V} = \int_{\mathcal{V}} \delta \underline{\sigma}^T \frac{\partial b(\underline{\sigma})}{\partial \underline{\sigma}} \, d\mathcal{V} = \int_{\mathcal{V}} \delta b(\underline{\sigma}) \, d\mathcal{V} = \delta \int_{\mathcal{V}} b(\underline{\sigma}) \, d\mathcal{V} = \delta B(\underline{\sigma}), \quad (7.70)$$

where the chain rule for derivatives was used. $B(\underline{\sigma})$ is the *total stress energy* of the body. The principle of complementary virtual work now writes

$$\delta B(\underline{\sigma}) = \int_{S_2} \hat{\mathbf{u}} \cdot \delta \mathbf{t} \, dS. \quad (7.71)$$

Next, the prescribed displacements are assumed to be derivable from a potential function

$$\hat{\mathbf{u}} = -\frac{\partial \chi(\mathbf{t})}{\partial \mathbf{t}}, \quad (7.72)$$

where $\chi(\mathbf{t})$ is the *potential of the prescribed displacements*. For instance, the potential of prescribed displacements is simply $\chi = -\hat{\mathbf{u}} \cdot \mathbf{t}$. It is important to note that potential functions do not exist for all types of prescribed displacements. Potential functions do not always exist for displacements that depend on the surface tractions, although such cases are not common in practice.

The right hand side of the principle of complementary virtual work, eq. (7.43), now becomes

$$\int_{S_2} \hat{\mathbf{u}} \cdot \delta \mathbf{t} \, dS = - \int_{S_2} \frac{\partial \chi}{\partial \mathbf{t}} \cdot \delta \mathbf{t} \, dS = - \int_{S_2} \delta \chi(\mathbf{t}) \, dS = - \delta \int_{S_2} \chi(\mathbf{t}) \, dS = -\delta X. \quad (7.73)$$

where $X(\mathbf{t}) = \int_{\mathcal{S}_2} \chi(\mathbf{t}) \, d\mathcal{S}$ is the total potential the prescribed displacements. Introducing this result into eq. (7.71) leads to

$$\delta B(\underline{\sigma}) = -\delta X(\mathbf{t}), \text{ or } \delta (B(\underline{\sigma}) + X(\mathbf{t})) = 0. \quad (7.74)$$

The *total complementary energy* of the body is now defined as

$$\Pi_C(\sigma) = B(\underline{\sigma}) + X(\mathbf{t}), \quad (7.75)$$

and it follows that

$$\delta \Pi_C(\sigma) = 0. \quad (7.76)$$

This statement can be interpreted as follows

Principle 4 (Principle of Minimum Complementary Energy) *Among all statically admissible stress fields, the actual stress field that corresponds to the compatible deformations of the body makes Π_C an absolute minimum.*

Eq. (7.76) only proves that for compatible deformations, the total complementary energy presents a stationary point. As discussed in section 7.1.1, the sign of the second variation $\delta^2 \Pi_C$ will determine whether the stationary point actually is a minimum. The first variation in Π_C is

$$\delta \Pi_C(\sigma) = \int_{\mathcal{V}} \sum_{i=1}^6 \frac{\partial b}{\partial \sigma_i} \delta \sigma_i \, d\mathcal{V} - \int_{\mathcal{S}_2} \hat{\mathbf{u}} \cdot \delta \mathbf{t} \, d\mathcal{S}, \quad (7.77)$$

and its second variation is then

$$\delta^2 \Pi_C(\sigma) = \int_{\mathcal{V}} \sum_{i,j=1}^6 \frac{\partial^2 b}{\partial \sigma_i \partial \sigma_j} \delta \sigma_i \delta \sigma_j \, d\mathcal{V}. \quad (7.78)$$

Just as for the strain energy density function, the stress energy density function must be a positive-definite function of the stress components, which implies $\sum_{i,j=1}^6 \partial^2 b / (\partial \sigma_i \partial \sigma_j) \delta \sigma_i \delta \sigma_j \geq 0$ for all $\delta \sigma_i$. Indeed, if the stress energy function were not positive-definite, stress states would exist that generate a negative stress energy, *i.e.* the elastic body would generate energy under stress, a situation that is physically impossible. It follows that $\delta^2 \Pi_C \geq 0$, and hence, Π_C present an absolute minimum at its stationary point.

If the principle of minimum complementary energy holds true, the complementary energy must present a stationary point, implying eq. (7.74). In turns, this equation implies the principle of complementary virtual work, in which the strains were expressed in terms of the stresses using constitutive laws of the form (7.68). If summary, the principle of minimum complementary energy implies the strain-displacements relationships of the problem expressed in terms of the stress field, which must satisfy equilibrium equations. Such equations are the Euler-Lagrange equations corresponding to the stationarity condition for the complementary energy, see section 7.1.3. The principle of minimum complementary energy implies the principle of complementary virtual work, but the principle of complementary virtual work only implies the principle of minimum complementary energy under restrictive assumptions on existence of a stress energy density function and of potential of the prescribed displacements.

7.2.8 The principle of least work

If the portion $\mathcal{S}_2$ of the outer surface of the body is rigidely held, $\hat{\mathbf{u}} = 0$, and the principle of minimum complementary energy reduces to $\delta B(\underline{\sigma}) = 0$, a statement known as the *principle of least work* that can be interpreted as follows

Principle 5 (Principle of Least Work) *Among all statically admissible stress fields, the actual stress field that corresponds to the compatible deformations of a rigidely held body makes the total stress energy an absolute minimum.*

7.2.9 Castigliano's theorem

Consider an elastic structure subjected to prescribed displacements u_i , $i = 1, 2, \dots, n$ at n points. The total complementary energy of the system is

$$\Pi_C = B - \sum_{i=1}^n u_i P_i, \quad (7.79)$$

where P_i are the required forces at the points where the displacements u_i are prescribed, projected in the direction of these displacements, as shown in fig. 7.5. B is the total complementary energy expressed in terms of the stress field which must be in equilibrium with the applied loads P_i , and hence, $B = B(P_i)$. The stationarity of the complementary energy then implies

$$\delta \Pi_C = \sum_{i=1}^n \frac{\partial B}{\partial P_i} \delta P_i - \sum_{i=1}^n u_i \delta P_i = \sum_{i=1}^n \left[\frac{\partial B}{\partial P_i} - u_i \right] \delta P_i = 0. \quad (7.80)$$

Since δP_i , $i = 1, 2, \dots, n$ are arbitrary variations, a valid choice is $\delta P_1 = 1$, $\delta P_i = 0$, $i = 2, \dots, n$. This implies $u_1 = \partial B / \partial P_1$. Similarly, selecting $\delta P_1 = 0$, $\delta P_2 = 1$, $\delta P_i = 0$, $i = 3, \dots, n$, yields $u_2 = \partial B / \partial P_2$. Collecting all results leads to

$$u_i = \frac{\partial B}{\partial P_i}, \quad i = 1, 2, \dots, n. \quad (7.81)$$

This statement can be interpreted as follows

Theorem 2 (Castigliano's theorem) *For an elastic structure, the displacement of the point of application of a force, projected in the direction of this force, equals the partial derivative of the complementary energy of the structure with respect to the force. The complementary energy of the structure must be expressed in terms of the stress field in equilibrium with the applied forces.*

Note that Castigliano's theorem is valid for elastic structures, whereas Clapeyron's theorem, theorem 1, only holds for linear elastic structures. Castigliano's theorem also remains valid if the structure is subjected to couples M_i , $i = 1, 2, \dots, n$. Let ϕ_i be the rotations at the point of application of the couples projected along the axis of the couple, then

$$\phi_i = \frac{\partial B}{\partial M_i}, \quad i = 1, 2, \dots, n. \quad (7.82)$$

Finally, consider two opposite forces sharing a common line of action, such as force P_3 and P_4 in fig. 7.5. Castigliano's theorem applied to both forces yields $u_3 = \partial B / \partial P_3$ and $u_4 = \partial B / \partial P_4$. Since the forces are acting in opposite direction, the relative displacement u of their points of applications become $u = \partial B / \partial P_3 + \partial B / \partial P_4$. If the two forces take a common value, $P_3 = P_4 = P$, $\partial B / \partial P_3 + \partial B / \partial P_4 = \partial B / \partial P$, and

$$u = \frac{\partial B}{\partial P}. \quad (7.83)$$

This statement can be interpreted as follows

Theorem 3 *If two opposite forces acting on an elastic structure share a common line of action, the relative displacement of their points of application, projected along their line of action, equals the partial derivative of the complementary energy of the structure with respect to the common value of the forces. The complementary of the structure must be expressed in terms of the stress field in equilibrium with the applied forces.*

The relative displacement is a positive quantity if the forces pull away from each other, and is negative in the opposite case.

7.3 Applications of Energy Theorems

In this section several application of the various energy theorems presented above will be discussed.

7.3.1 Applications of Clapeyron's theorem

Simply supported beam with concentrated load

Consider a simply supported, uniform beam of length L subjected to a concentrated load P acting at a distance αL from the end supports, as depicted in fig. 3.21. The bending moment distribution was obtained in example 2 of section 3.5.9, see eq. (3.71). The curvature distribution is then readily obtained as

$$\kappa_3(x_1) = \frac{M_3^c}{I_{33}^c} = \frac{PL}{I_{33}^c} \begin{cases} -(1-\alpha)\eta; & 0 \leq \eta \leq \alpha \\ -\alpha(1-\eta); & \alpha < \eta \leq 1 \end{cases}.$$

The total strain energy stored in the beam is then obtained from eq. (3.66)

$$A(\kappa_3) = \frac{1}{2} \int_0^L I_{33}^c \kappa_3^2 dx_1 = \frac{I_{33}^c}{2} \frac{P^2 L^2}{I_{33}^{c2}} \left[\int_0^\alpha (1-\alpha)^2 \eta^2 L d\eta + \int_\alpha^1 \alpha^2 (1-\eta)^2 L d\eta \right].$$

Performing the integrations then leads to

$$A(\kappa_3) = \frac{P^2 L^3}{2 I_{33}^c} \left[(1-\alpha)^2 \frac{\alpha^3}{3} + \alpha^2 \frac{(1-\alpha)^3}{3} \right] = \frac{P^2 L^3}{6 I_{33}^c} \alpha^2 (1-\alpha)^2.$$

Clapeyron's theorem, eq. (7.64) now yields

$$u = \frac{2A}{P} = \frac{PL^3}{3I_{33}^c} \alpha^2 (1 - \alpha)^2. \quad (7.84)$$

This result can be validated by evaluating the transverse deflection under the applied load, $U_2(\alpha)$ from the exact solution, eq. (3.70). Identical results are obtained.

Clapeyron's theorem yields the deflection under the load in a very expeditious manner. In contrast, the classical approach of section 3.5.9 requires the solution of the governing differential equation of the problem and yields the distributions of displacements, bending moments, and shear forces over the span of the beam. More detailed information is obtained, but at a higher cost.

7.3.2 Applications of Castigliano's theorem

Simply supported beam with concentrated load

Consider a simply supported, uniform beam of length L subjected to a concentrated load P acting at a distance αL from the end supports, as depicted in fig. 3.21. Castigliano's theorem will be used to evaluate the displacement of the point of application of the load P . The complementary energy of the beam is given by eq. (3.67), and Castigliano's theorem, theorem 2, then yields

$$u = \frac{\partial B}{\partial P} = \frac{\partial}{\partial P} \int_0^L \frac{M_3^{c2}}{2I_{33}^c} dx_1 = \int_0^L \frac{M_3^c}{I_{33}^c} \frac{\partial M_3^c}{\partial P} dx_1.$$

The bending moment distribution was obtained in example 2 of section 3.5.9, see eq. (3.71). It follows that

$$u = \frac{PL^2}{I_{33}^c} \left[\int_0^\alpha (1 - \alpha)^2 \eta^2 L d\eta + \int_\alpha^1 \alpha^2 (1 - \eta)^2 L d\eta \right].$$

Performing the integrations then leads to

$$u = \frac{PL^3}{I_{33}^c} \left[(1 - \alpha)^2 \frac{\alpha^3}{3} + \alpha^2 \frac{(1 - \alpha)^3}{3} \right] = \frac{PL^3}{3I_{33}^c} \alpha^2 (1 - \alpha)^2.$$

This result can be validated by evaluating the transverse deflection under the applied load, $U_2(\alpha)$ from the exact solution, eq. (3.70). It can also be compared to the result obtained from the application of Clapeyron's theorem, eq. (7.84). Identical results are obtained for all three approaches.

Tip deflection and slope of a cantilevered beam

Consider the cantilevered beam subjected to a tip load P and a tip bending moment Q shown in fig 7.7. A free body diagram of a section of the beam extending from a station at location x_1 to the tip of the beam yields the bending moment distribution

$$M_3^c = Q + P(L - x_1).$$



Figure 7.7: Cantilevered beam under tip force and moment.

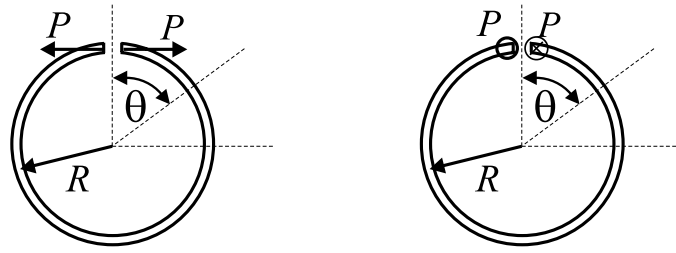


Figure 7.8: Ring subjected an internal forces acting in the plane of the ring, left figure, and out of the plane of the ring, right figure.

The tip deflection u and rotation ϕ of the beam are given by Castigliano's theorem as

$$u = \frac{\partial B}{\partial P}, \text{ and } \phi = \frac{\partial B}{\partial Q},$$

respectively. It follows that

$$u = \int_0^L \frac{M_3^c}{I_{33}^c} \frac{\partial M_3^c}{\partial P} dx_1 = \frac{1}{I_{33}^c} \int_0^L [Q + P(L - x_1)] (L - x_1) dx_1 = \frac{PL^3}{3I_{33}^c} + \frac{QL^2}{2I_{33}^c},$$

and

$$\phi = \int_0^L \frac{M_3^c}{I_{33}^c} \frac{\partial M_3^c}{\partial Q} dx_1 = \frac{1}{I_{33}^c} \int_0^L [Q + P(L - x_1)] dx_1 = \frac{PL^2}{2I_{33}^c} + \frac{QL}{I_{33}^c}.$$

Ring under internal forces

Consider the circular ring of radius R shown in fig. 7.8. The ring is cut at one location where two opposite forces of equal magnitude P are applied along the same line of action located in the plane of the ring (left figure). The relative displacement u of the points of application of the two forces will be computed with the help of Castigliano's theorem as

$$u = \frac{\partial B}{\partial P} = \int_0^{2\pi} \frac{M_3^c}{I_{33}^c} \frac{\partial M_3^c}{\partial P} R d\theta + \int_0^{2\pi} \frac{F_1}{S} \frac{\partial F_1}{\partial P} R d\theta,$$

where $M_3^c = PR(1 - \cos \theta)$ is the bending moment distribution in the ring and $F_1 = -P \cos \theta$ the axial force distribution. The relative displacement becomes

$$u = \frac{PR^3}{I_{33}^c} \int_0^{2\pi} (1 - \cos \theta)^2 d\theta + \frac{PR}{S} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{3\pi PR^3}{I_{33}^c} + \frac{\pi PR}{S}.$$

If the ring has a rectangular cross-section of thickness h and depth b ,

$$u = \frac{36\pi PR^3}{Eb h^3} + \frac{\pi PR}{Eb h} = \frac{\pi PR}{Eb h} \left[36\left(\frac{R}{h}\right)^2 + 1 \right].$$

For a thin ring, $R/h \gg 1$, and the first term becomes dominant, implying that bending of the ring is the principal contributor to the relative displacement of the points of application of the two forces.

Consider once again the circular ring with a cut shown in fig. 7.8 but subjected now to two opposite forces of equal magnitude P applied along the same line of action *normal to the plane of the ring* (right figure). With this new type of loading, bending and torsion will develop in the ring and Castigliano's theorem implies

$$u = \frac{\partial B}{\partial P} = \int_0^{2\pi} \frac{M_3^c}{I_{33}^c} \frac{\partial M_3^c}{\partial P} R d\theta + \int_0^{2\pi} \frac{M_1}{J} \frac{\partial M_1}{\partial P} R d\theta,$$

where $M_3^c = PR \sin \theta$ is the bending moment distribution and $M_1 = PR(1 - \cos \theta)$ the axial force distribution. Note that u is now the relative displacement of the points of application of the forces projected along their line of action, *i.e.* measured in the direction perpendicular to the plane of the ring. This displacement becomes

$$u = \frac{PR^3}{I_{33}^c} \int_0^{2\pi} \sin^2 \theta d\theta + \frac{PR^3}{J} \int_0^{2\pi} (1 - \cos \theta)^2 R d\theta = \frac{\pi PR^3}{I_{33}^c} + \frac{3\pi PR^3}{J}.$$

If the ring has a circular cross-section of radius a , and is made of a linear elastic, homogeneous material ($G = E/(2(1 + \nu))$)

$$u = \frac{4PR^3}{Ea^4} [1 + 3(1 + \nu)].$$

Note that the torsion in the ring contributes $[3(1 + \nu)]/[1 + 3(1 + \nu)] \approx 80\%$ of the total relative displacement.

Simply supported beam under a uniform load

Castigliano's theorem also allows the evaluation of displacements at locations where no force is applied by using the following artifact. At the desired location, a fictitious force $\mathcal{P}$ is applied, and the total stress energy including this fictitious force is computed. The desired displacement is then

$$u = \left. \frac{\partial B}{\partial \mathcal{P}} \right|_{\mathcal{P}=0}$$

where the derivative is computed first, then $\mathcal{P} = 0$ is introduced in the final result. This procedure will be illustrated by computing the transverse displacement distribution for the simply supported, uniform beam of length L subjected to a uniform transverse loading p_0 , as depicted in fig. 3.20. A fictitious force $\mathcal{P}$ is applied at a location αL from the left support. The bending moment distribution is then found by superposing the moment due to the

distributed load p_0 , eq. (3.69), and the moment due to the fictitious force $\mathcal{P}$, eq. (3.71) to find

$$M_3^c(x_1) = \begin{cases} \frac{p_0 L^2}{2} \eta(1 - \eta) - \mathcal{P}L(1 - \alpha)\eta; & 0 \leq \eta \leq \alpha \\ \frac{p_0 L^2}{2} \eta(1 - \eta) - \mathcal{P}L\alpha(1 - \eta); & \alpha < \eta \leq 1 \end{cases}.$$

The displacement at the point of application of the fictitious is now

$$u(\alpha) = \frac{\partial}{\partial \mathcal{P}} \int_0^L \frac{M_3^c}{2I_{33}^c} dx_1 \Big|_{\mathcal{P}=0} = \int_0^L \frac{M_3^c}{I_{33}^c} \Big|_{\mathcal{P}=0} \frac{\partial M_3^c}{\partial \mathcal{P}} dx_1.$$

Introducing the bending moment distribution then yields

$$u(\alpha) = \int_0^\alpha \frac{p_0 L^4}{2I_{33}^c} \eta(1 - \eta)(1 - \alpha)\eta d\eta + \int_\alpha^1 \frac{p_0 L^4}{2I_{33}^c} \eta(1 - \eta)\alpha(1 - \eta) d\eta.$$

Performing the integrations leads to

$$u(\alpha) = \frac{p_0 L^4}{2I_{33}^c} \left[(1 - \alpha) \left(\frac{\alpha^3}{3} - \frac{\alpha^4}{4} \right) + \alpha \left[\frac{(1 - \alpha)^3}{3} - \frac{(1 - \alpha)^4}{4} \right] \right].$$

After simplification, the transverse displacement distribution becomes

$$u(\alpha) = \frac{p_0 L^4}{24I_{33}^c} \alpha(1 - \alpha)(1 + \alpha - \alpha^2) = \frac{p_0 L^4}{24I_{33}^c} (\alpha - 2\alpha^3 + \alpha^4).$$

This result is identical to the transverse displacement distribution found by integrating the governing differential equations, eq. (3.68).

Three bar hyperstatic system

Castigliano's theorem is particularly powerful when dealing with hyperstatic system. Consider for instance, the simple three bar truss shown in fig. 2.2. The classical approach to this type of problem is to compute the deformations of the system in terms of the loads in the bars. The compatibility conditions then provide additional relationships to evaluate the unknown forces in the bars. In other words, a complete solution of the problem is required to evaluate the internal loads.

With Castigliano's theorem, an alternate approach becomes possible. The system is cut at a specific location and two opposite forces P of equal magnitude acting along a common line of action are then added at the location of the cut. The relative displacement of their point of application is $u = \partial B / \partial P$. In the actual system, no such cut exists implying the vanishing of the relative displacement

$$u = \frac{\partial B}{\partial P} = 0.$$

This relationship provides an equation to evaluate the force P .

Considering fig. 2.2, the three bar truss system is cut at point B and F_B is the magnitude of the forces acting at this cut, *i.e.* the force in the cut bar. The complementary energy of the system is

$$B = 2 \int_0^{L/\cos\Theta} \frac{F_A^2}{2E_A\Omega_A} dx + \int_0^L \frac{F_B^2}{2E_B\Omega_B} dx = \frac{F_A^2}{E_A\Omega_A} \frac{L}{\cos\Theta} + \frac{F_B^2}{2E_B\Omega_B} L,$$

where the notation of section 2.3 were used. Vertical equilibrium yields $F_B + 2F_A \cos\Theta = P$. The two unknown forces F_B and F_A cannot be solely determined from equilibrium considerations. The complementary energy is now expressed in terms on the unknown force at the cut

$$B = \frac{L}{E_A\Omega_A} \frac{1}{4\cos^3\Theta} (P - F_B)^2 + \frac{L}{2E_B\Omega_B} F_B^2,$$

and $\partial B/\partial F_B = 0$ then yields

$$-\frac{2L}{E_A\Omega_A} \frac{1}{4\cos^3\Theta} (P - F_B) + \frac{2L}{2E_B\Omega_B} F_B = 0.$$

This equation can be solved for F_B to find

$$F_B = \frac{P}{1 + 2\cos^3\Theta \frac{E_A\Omega_A}{E_B\Omega_B}},$$

a result identical to that found in section 2.3.

Cantilevered beam with simple support

Consider now the hyperstatic problem consisting of a cantilevered beam with an additional support at location $x_1 = \alpha L$, as depicted in fig. 7.9. A uniform loading p_0 is applied to the beam. Following the procedure outlined in the previous section, the system is cut at the simple support and let F be the magnitude of the two opposite forces acting in the vertical direction at this cut. The bending moment distribution in the beam can now be found from equilibrium considerations

$$M_3^c = \begin{cases} \frac{p_0 L^2}{2} (1 - \eta)^2 - FL(\alpha - \eta); & 0 \leq \eta \leq \alpha \\ \frac{p_0 L^2}{2} (1 - \eta)^2; & \alpha < \eta \leq 1 \end{cases},$$

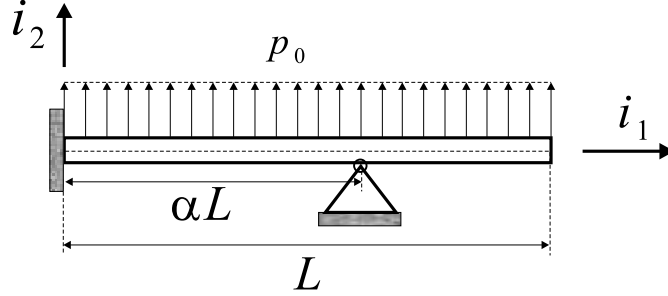
where $\eta = x_1/L$ is the nondimensional span-wise variable.

In the actual system, the relative displacement at the cut must vanish

$$0 = \frac{\partial B}{\partial F} = \int_0^L \frac{M_3^c}{I_{33}^c} \frac{\partial M_3^c}{\partial F} dx_1 = \int_0^\alpha \left[\frac{p_0 L^2}{2} (1 - \eta)^2 - FL(\alpha - \eta) \right] [-L(\alpha - \eta)] L d\eta.$$

This equation can be solved to determine the unknown reaction force

$$F = \frac{p_0 L}{8} \frac{6 - 4\alpha + \alpha^2}{\alpha}.$$

Figure 7.9: Cantilevered beam with a simple support at location αL .

A free body diagram of the entire beam reveal $Q = p_0 L^2/2 - \alpha FL$, where Q is the reaction moment at the clamped root of the beam. Hence,

$$Q = -\frac{p_0 L^2}{8}(\alpha^2 - 4\alpha + 2).$$

The same problem could have been solved by performing a simple cut at the root of the beam. The clamp is replaced by a simple support and Q is the magnitude of the two opposite moments acting at that location. The bending moment distribution in the beam is once again found from equilibrium considerations

$$M_3^c = \begin{cases} \frac{p_0 L^2}{2} \left[\eta^2 + \left(\frac{1}{\alpha} - 2 \right) \eta \right] + Q \left(1 - \frac{\eta}{\alpha} \right); & 0 \leq \eta \leq \alpha \\ \frac{p_0 L^2}{2} (1 - \eta)^2; & \alpha < \eta \leq 1 \end{cases},$$

The relative rotation at the cut must, of course, vanish

$$0 = \frac{\partial B}{\partial Q} = \int_0^L \frac{M_3^c}{I_{33}^c} \frac{\partial M_3^c}{\partial Q} dx_1 = \int_0^\alpha \left\{ \frac{p_0 L^2}{2} \left[\eta^2 + \left(\frac{1}{\alpha} - 2 \right) \eta \right] + Q \left(1 - \frac{\eta}{\alpha} \right) \right\} \left(1 - \frac{\eta}{\alpha} \right) L d\eta.$$

This equation can be solved to determine the unknown reaction moment to find $Q = -p_0 L^2(\alpha^2 - 4\alpha + 2)/8$, a result identical to that found by performing the cut at the simple support.

Simply supported beam with two elastic spring supports

A simply supported beam of span L is supported by two spring of stiffness k located at stations $x_1 = \alpha L$ and $(1 - \alpha)L$, and is subjected to a uniform transverse loading p_0 , as depicted in fig. 3.23. This again is a hyperstatic problem and a cut is performed at the ground connection of the left spring. Let F be the magnitude of the two opposite forces acting in the vertical direction at this cut. The bending moment distribution is readily found from simple equilibrium considerations

$$M_3^c = \begin{cases} -\frac{p_0 L^2}{2} \eta(1 - \eta) - FL\eta; & 0 \leq \eta \leq \alpha \\ -\frac{p_0 L^2}{2} \eta(1 - \eta) - FL\alpha; & \alpha < \eta \leq 1/2 \end{cases}.$$

The complementary energy of the system is

$$B = 2 \left[\int_0^{L/2} \frac{M_3^2}{2I_{33}^c} dx_1 + \frac{F^2}{2k} \right],$$

where the second term represents the stress energy stored in the elastic spring. The symmetry of the system was taken into account to evaluate the energy. The relative displacement at the cut vanishes if

$$\frac{\partial B}{\partial F} = 2 \left[\int_0^{L/2} \frac{M_3^c}{I_{33}^c} \frac{\partial M_3^c}{\partial F} dx_1 + \frac{F}{k} \right] = 0.$$

Introducing the bending moment distribution then yields

$$\int_0^\alpha \left[-\frac{p_0 L^2}{2I_{33}^c} \eta(1-\eta) - \frac{FL}{I_{33}^c} \eta \right] (-L\eta) L d\eta + \int_\alpha^{1/2} \left[-\frac{p_0 L^2}{2I_{33}^c} \eta(1-\eta) - \frac{FL}{I_{33}^c} \alpha \right] (-L\alpha) L d\eta + \frac{F}{k} = 0.$$

This equation can be solved for the unknown force F acting in the spring. After integration this becomes

$$\frac{p_0 L^4}{24I_{33}^c} (4\alpha^3 - 3\alpha^4) + \frac{FL^3}{I_{33}^c} \frac{\alpha^3}{3} + \frac{p_0 L^4}{24I_{33}^c} (\alpha - 6\alpha^3 + 4\alpha^4) + \frac{FL^3}{I_{33}^c} \left(\frac{\alpha^2}{2} - \alpha^3 \right) + \frac{F}{k} = 0.$$

The reaction at the spring support is then

$$F = -\frac{p_0 L}{4} \frac{k^*(\alpha - 2\alpha^3 + \alpha^4)}{6 + k^*(3\alpha^2 - 4\alpha^3)},$$

where $k^* = kL^3/I_{33}^c$ is the nondimensional spring stiffness constant. The same result was obtained using the classical approach, see eq. (3.73). The difference in sign between the two results reflects the different sign conventions for the reactions force.

7.3.3 Problems

Problem 7.1

Consider the uniform, semi-circular spring with a rigid arm, as shown in fig. 7.10. The spring is made of a linear elastic material. A load P is acting at the tip of the rigid arm, in the plane of the spring and its orientation is arbitrary. Prove that

1. The displacement δ of point O is in the direction of the applied load *for any arbitrary orientation* of P .
2. The spring constant $k = P/\delta$ is independent of the orientation of the load P .

Hint: At first, study the behavior of the spring under a horizontal force H . Next, turn to a vertical force V . The behavior of the system under a general loading is then obtained by invoking the principle of superposition for a linear system. You can assume that bending deformations only contribute to the spring internal energy.

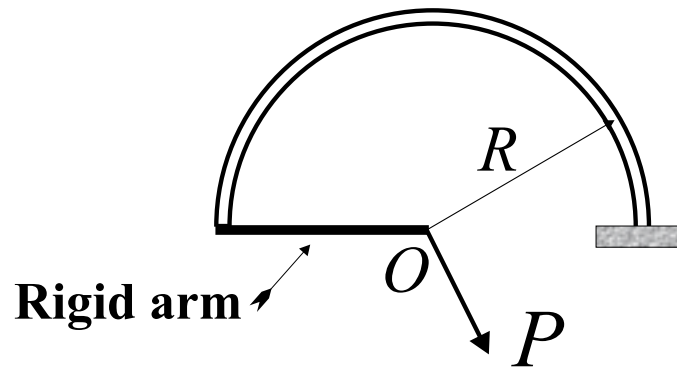
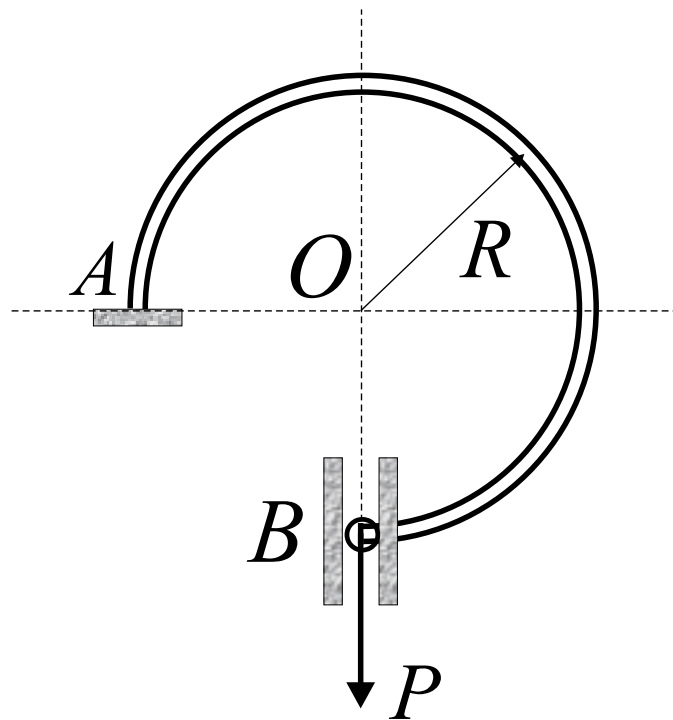


Figure 7.10: Semi-circular spring with a rigid arm.

Figure 7.11: Uniform spring under a vertical load P .

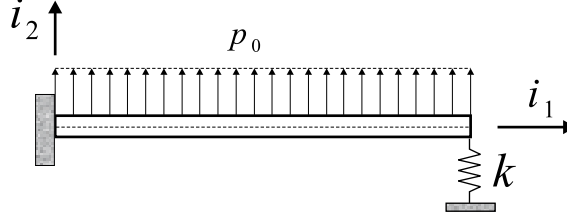


Figure 7.12: Cantilevered beam with a tip spring.

Problem 7.2

The uniform spring shown in fig. 7.11 is clamped at point A and constrained to move in the sole vertical direction at point B where it is subjected to an applied vertical load P .

1. Find the displacement δ in the direction of the applied load.
2. Find the horizontal reaction Q at point B .
3. Find the equivalent spring constant $k = P/\delta$.

Assume that only bending deformations are significant.

Problem 7.3

Consider the cantilevered beam shown in fig. 7.12. It is subjected to a uniform transverse loading p_0 and has a tip spring of stiffness k . Find the force in the tip spring using Castigliano's theorem.

7.3.4 Reciprocity Theorem

Consider here again the linear elastic structure subjected to concentrated loads P_i , $i = 1, 2, \dots, n$, as depicted in fig. 7.5. If the displacements of the points of application of these loads projected along their lines of action are denoted u_i , Clapeyron's theorem, eq. (7.64), implies $A_1 = \sum_{i=1}^n P_i u_i / 2$, where A_1 is the total strain energy of the system under this loading condition, denoted *state 1*. Let the magnitude the of applied loads be changed to P'_i and the corresponding projected displacements be u'_i . In this new state, denoted *state 2*, the points of application of the loads and their lines of action are identical to those in *state 1*. The total potential energy in *state 2* is $A_2 = \sum_{i=1}^n P'_i u'_i / 2$. When going from *state 1* to *state 2*, the work done by the applied loads equals the change in total strain energy

$$A_2 - A_1 = \sum_{i=1}^n \frac{P'_i u'_i}{2} - \sum_{i=1}^n \frac{P_i u_i}{2}. \quad (7.85)$$

This difference can be computed in an alternate manner. It is possible to go from *state 1* to *state 2* by gradually adding to the force P_i of *state 1* the forces $P'_i - P_i$ with unchanged lines of action. During this transition, the work done by the constant forces P_i is $\sum_{i=1}^n P_i (u'_i - u_i)$, whereas the work done by the gradually increasing forces $P'_i - P_i$ is $\sum_{i=1}^n (P'_i - P_i)(u'_i - u_i) / 2$.

The change in strain energy between the two states is then equal to the work done by the applied force, *i.e.*

$$A_2 - A_1 = \sum_{i=1}^n P_i(u'_i - u_i) + \sum_{i=1}^n \frac{(P'_i - P_i)(u'_i - u_i)}{2} = \frac{1}{2} \sum_{i=1}^n (P'_i + P_i)(u'_i - u_i). \quad (7.86)$$

Comparing eqs. (7.85) and (7.86) then yields

$$\sum_{i=1}^n P_i u'_i = \sum_{i=1}^n P'_i u_i. \quad (7.87)$$

This result be interpreted as follows

Theorem 4 (Reciprocity theorem or Betti's theorem) *Consider a linear elastic body subjected to two loading states characterized by loads of different magnitudes but identical points of applications and lines of action. The sum of the product of the loads in one state by the projected displacements of the other is the same as that obtained when the two states are interchanged.*

Since Betti's theorem is a direct consequence of Clapeyron's theorem, theorem 1, both theorems are valid for the same loading cases. The loadings defining *states 1* and *2* can involve one or more of the following 1) a concentrated load and the projected displacement of its point of application, 2) a couple and the projected rotation at its point of application, 3) two opposite forces of identical magnitude sharing a common line of action and the projected relative displacement of their points of application, and 4) two opposite couples of identical magnitude sharing a common line of action and the projected relative rotation at their points of application.

7.3.5 Maxwell's Theorem

Let the simply supported beam depicted in fig. 7.13 be subjected to two loading states. *State 1* consists of a load P_1 at point 1, whereas *state 2* consists of a load P_2 at point 2. For *state 1*, the displacements at points 1 and 2 will be denoted u_{11} and u_{21} , respectively. Similarly, the corresponding displacements for *state 2* are u_{12} and u_{22} , respectively.

If the structure is made of a linear elastic material, Betti's theorem, theorem 4, is applicable and implies

$$P_1 u_{12} = P_2 u_{21}, \text{ or } \frac{u_{12}}{P_2} = \frac{u_{21}}{P_1}. \quad (7.88)$$

The *influence coefficient* is defined as the displacement at a point due to a unit load at another point. For instance, the influence coefficient η_{12} gives the displacement at point 1 due to a unit load at point 2. Clearly, $\eta_{12} = u_{12}/P_2$ and η_{21} , the displacement at point 2 due to a unit load applied at point 1, is $\eta_{21} = u_{21}/P_1$. Eq. (7.88) becomes

$$\eta_{12} = \eta_{21}. \quad (7.89)$$

This result be interpreted as follows

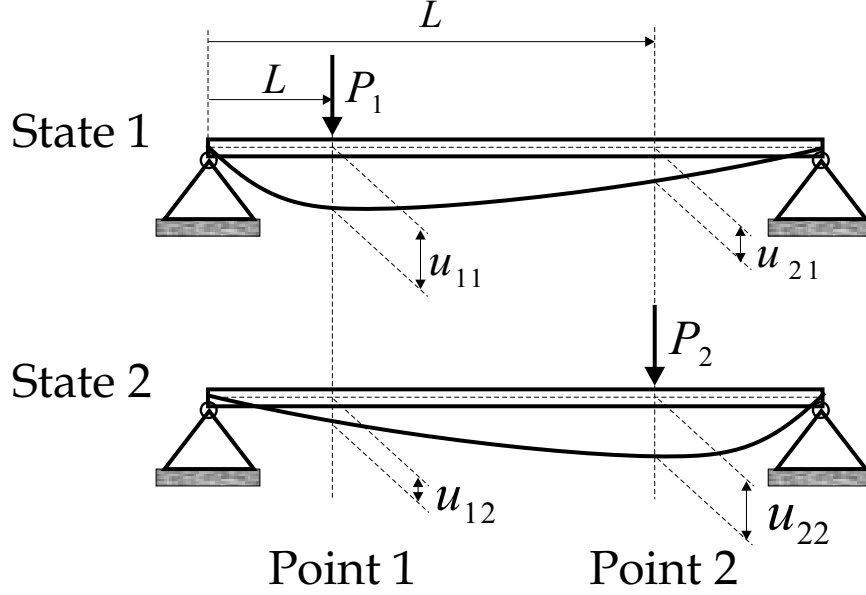


Figure 7.13: Simply supported beam under two loading states.

Theorem 5 (Maxwell's theorem) *For a linear elastic structure the influence coefficient of point 1 on point 2 equals that of point 2 on point 1, for any choice of points 1 and 2.*

Maxwell's theorem clearly is a simple corollary of Betti's theorem, and applies to all the loading conditions this latter theorem is valid for. Hence, the concept of influence coefficient should be understood as the projected displacement or rotation at a point, relative displacement of two points, or relative rotation at two points due to any of the following four loading types applied at another location: a concentrated load or couple, or two opposite forces of identical magnitude sharing a common line, or two opposite couples of identical magnitude sharing a common line of action.

Simply supported beam

Consider a simply supported, uniform beam of length L as shown in fig. 7.13. Let points 1 and 2 be located at distances αL and βL from the left end support. The influence coefficient η_{12} can be evaluated from the exact solution of the problem given in section 3.5.9. η_{12} is the displacement at point 1 when the beam is subjected to a unit load at point 2. Eq. (3.70) gives this displacement as

$$\eta_{12} = \frac{L^3}{6I_{33}^c} [-(1-\beta)\alpha^3 + \beta(2-\beta)(1-\beta)\alpha] = \frac{L^3}{6I_{33}^c} [-\alpha^3 + \alpha^3\beta + 2\alpha\beta - 3\alpha\beta^2 + \alpha\beta^3].$$

The following values were used: $P = 1$ since a unit load is applied, $\alpha = \beta$ since this load is applied at $\eta = \beta$, and $\eta = \alpha$ to find the displacement $U_2(\alpha)$.

The influence coefficient η_{21} , corresponding to the displacement at point 2 when the beam

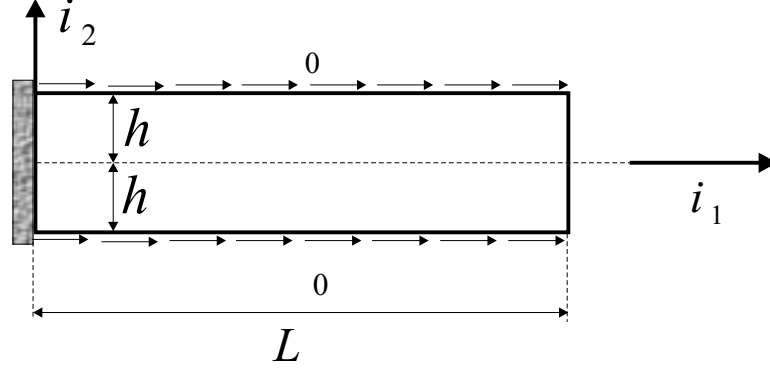


Figure 7.14: Upper skin of a box beam subjected to edge shear forces.

is subjected to a unit load at point 1 is evaluated in a similar manner, to find

$$\eta_{21} = \frac{L^3}{6I_{33}^c} [\alpha(\beta^3 - 3\beta^2) + \alpha(2 + \alpha^2)\beta - \alpha^3] = \frac{L^3}{6I_{33}^c} [\alpha\beta^3 - 3\alpha\beta^2 + 2\alpha\beta + \alpha^3\beta - \alpha^3].$$

The following values were used: $P = 1$ and $\eta = \beta$ to find the displacement $U_2(\beta)$. It is clear that $\eta_{12} = \eta_{21}$, in accordance with Maxwell's theorem.

7.4 Applications of Variational and Energy Principles

7.4.1 The shear lag problem

Consider a thin-walled, rectangular box beam clamped at its root. The beam presents stiffeners at its four corners and is subjected to bending. According to Euler-Bernoulli beam theory, the stress distribution in the upper flange of the beam consists of a uniform distribution of tensile axial stress, whereas compressive stresses arise in the lower flange. Near the built-in edge of the beam, the axial stress distribution might be markedly different due to a phenomenon called *shear lag*.

A crude analysis of this phenomenon is presented in this section. The upper flange of the box beam is approximated by a rectangular panel subjected to distributed edge forces N_0 , as depicted in fig. 7.14. The width of the panel is $2h$, its thickness t , and its length L ; N_0 represents the loads the corner stiffeners apply on the upper flange of the box beam. The displacement field in the panel is assumed to take the following form

$$u_1(\eta, \zeta) = a(\eta) + b(\eta)(\zeta^2 - \frac{1}{3}); \quad u_2(\eta, \zeta) = 0, \quad (7.90)$$

where $\eta = x_1/L$ is the nondimensional variable along the span of the panel and $\zeta = x_2/h$ that across its width. The axial displacement distribution is illustrated in fig. 7.15 and consists of a uniform distribution across the width of the panel with a magnitude $a(\eta)$ and of a parabolic distribution across the width $(\zeta^2 - 1/3)$ with a magnitude $b(\eta)$. The first term will give rise to uniform displacements, strains, and stresses as predicted by beam theory.

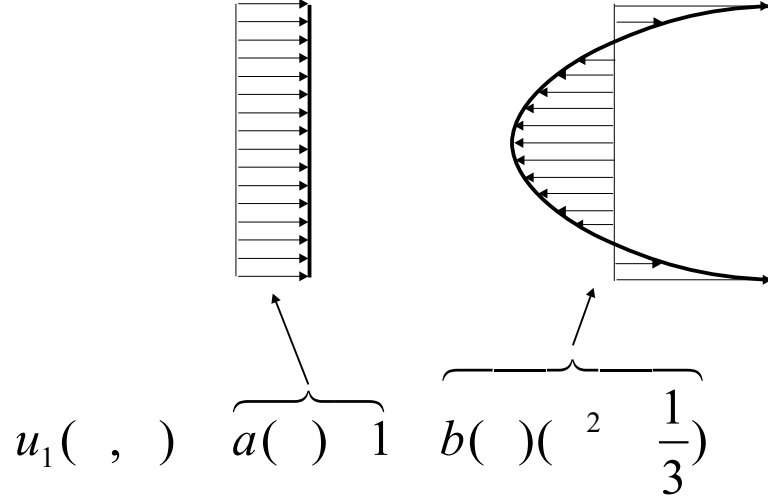


Figure 7.15: The distribution of axial displacement.

The second term will result in an uneven stress distribution across the width of the panel and characterizes the magnitude of the shear lag effect. Since the panel is clamped at the root, the root axial displacement should vanish, and $a(0) = b(0) = 0$.

The strain field is obtained from the assumed displacement field as

$$\varepsilon_1(\eta, \zeta) = \frac{a'}{L} + \frac{b'}{L}(\zeta^2 - \frac{1}{3}); \quad \varepsilon_2(\eta, \zeta) = 0; \quad \gamma_{12}(\eta, \zeta) = \frac{b}{h}2\zeta, \quad (7.91)$$

where the notation $(\cdot)'$ indicates a derivative with respect to η . The panel is assumed to be made of a linear elastic material in plane stress state. The strain energy density function is then given by eq. (7.59), with the material stiffness matrix of eq. (7.60). Furthermore, the stress in the transverse direction is assumed to be much smaller than the axial stress, $\sigma_2 \ll \sigma_1$, and the constitutive laws then reduce to

$$\sigma_1 = E\varepsilon_1 = E \left[\frac{a'}{L} + \frac{b'}{L}(\zeta^2 - \frac{1}{3}) \right]; \quad \sigma_2 \approx 0; \quad \tau_{12} = G\gamma_{12} = G\frac{b}{h}2\zeta. \quad (7.92)$$

The strain energy in the panel now reduces to

$$A = \int_0^L \int_{-h}^{+h} \int_{-t/2}^{t/2} \frac{1}{2}(E\varepsilon_1^2 + G\gamma_{12}^2) dx_1 dx_2 dx_3 = \frac{Lht}{2} \int_0^1 \int_{-1}^{+1} (E\varepsilon_1^2 + G\gamma_{12}^2) d\eta d\zeta.$$

Introducing the strain distributions, eq. (7.91), and reordering the term leads to

$$\begin{aligned} A &= \frac{Lht}{2} \int_0^1 \left\{ \frac{Ea'^2}{L^2} \left[\int_{-1}^{+1} d\zeta \right] + \frac{Eb'^2}{L^2} \left[\int_{-1}^{+1} (\zeta^2 - \frac{1}{3})^2 d\zeta \right] \right. \\ &\quad \left. + 2\frac{Ea'b'}{L^2} \left[\int_{-1}^{+1} (\zeta^2 - \frac{1}{3}) d\zeta \right] + \frac{4Gb^2}{h^2} \left[\int_{-1}^{+1} \zeta^2 d\zeta \right] \right\} d\eta \\ &= \frac{Eht}{L} \int_0^1 \left[a'^2 + \frac{4}{45}b'^2 + \frac{4}{3} \left(\frac{L}{h} \right)^2 \left(\frac{G}{E} \right) b^2 \right] d\eta. \end{aligned}$$

The potential of the applied loads consists of the potential of the shear loads applied along the top and bottom edges of the panel,

$$\Phi = \int_{t/2}^{t/2} \int_0^L [\tau_0 u_1(x_1, x_2 = -h) + \tau_0 u_1(x_1, x_2 = +h)] dx_1 = 2tL \int_0^1 \tau_0 \left(a + \frac{2b}{3}\right) d\eta,$$

and the total potential energy becomes

$$\Pi = \frac{Eht}{L} \int_0^1 \left[a'^2 + \frac{4}{45} b'^2 + \frac{4}{3} \left(\frac{L}{h}\right)^2 \left(\frac{G}{E}\right) b^2 - \frac{2\tau_0 L^2}{Eh} \left(a + \frac{2b}{3}\right) \right] d\eta.$$

The principle of minimum total potential energy requires Π to be a stationary quantity, $\delta\Pi = 0$, this implies

$$\delta\Pi = \frac{2Eht}{L} \int_0^1 \left[a' \delta a' + \frac{4}{45} b' \delta b' + \frac{4}{3} \left(\frac{L}{h}\right)^2 \left(\frac{G}{E}\right) b \delta b - \frac{\tau_0 L^2}{Eh} (\delta a + \frac{2}{3} \delta b) \right] d\eta = 0.$$

Integration by parts is performed on the first two terms, then regrouping yields

$$\begin{aligned} \int_0^1 \left\{ \delta a \left[-a'' - \frac{\tau_0 L^2}{Eh} \right] + \delta b \left[-\frac{4}{45} b'' + \frac{4}{3} \left(\frac{L}{h}\right)^2 \left(\frac{G}{E}\right) b - \frac{2\tau_0 L^2}{3Eh} \right] \right\} d\eta \\ + [a' \delta a]_0^1 + \left[\frac{4}{45} b' \delta b \right]_0^1 = 0. \end{aligned}$$

The stationarity condition of Π requires $\delta\Pi = 0$ for all arbitrary variations δa and δb . Consequently, the bracketed term must vanish, resulting in the following differential equations for a

$$a'' = -\frac{\tau_0 L^2}{Eh}; \quad a(0) = 0, \quad a'(1) = 0,$$

and b

$$b'' - \mu^2 b = -\frac{15\tau_0 L^2}{2Eh}; \quad b(0) = 0, \quad b'(1) = 0,$$

where

$$\mu^2 = 15 \left(\frac{L}{h}\right)^2 \left(\frac{G}{E}\right). \quad (7.93)$$

These second order differential equations can be solved to obtain the axial displacement as

$$\frac{Ehu_1}{L^2\tau_0} = \eta - \frac{1}{2}\eta^2 + \frac{15}{2\mu^2} \left[1 - \frac{\cosh \mu(1-\eta)}{\cosh \mu} \right] \left(\zeta^2 - \frac{1}{3} \right). \quad (7.94)$$

The first two terms of this result, $\eta - \eta^2/2$ represent the axial displacement of the panel as if it were a beam of axial stiffness $S = E2ht$ subjected to a uniform axial load $p_0 = 2\tau_0 t$, see eq. (3.35). The second term represents the shear lag effect that is controlled by the

nondimensional parameter μ . The stress distribution in the panel is then obtained from the constitutive relationships, eq. (7.92)

$$\frac{h\sigma_1}{L\tau_0} = 1 - \eta + \frac{15}{2\mu} \frac{\sinh \mu(1 - \eta)}{\cosh \mu} \left(\zeta^2 - \frac{1}{3} \right); \quad (7.95)$$

$$\frac{\tau_{12}}{\tau_0} = \left[1 - \frac{\cosh \mu(1 - \eta)}{\cosh \mu} \right] \zeta. \quad (7.96)$$

Consider an aluminum panel with an aspect ratio $L = 8h$. The parameter $\mu = 19.2$, since $E/G = 2(1 + \nu) = 2.6$ for a homogeneous, isotropic material with a Poisson's ratio $\nu = 0.3$. On the other hand, for a panel with the same aspect ratio, but made of medium modulus Graphite fiber reinforced Epoxy matrix material, $\mu = 6.41$, since $E = 140 \text{ GPa}$ and $G = 6 \text{ GPa}$ for such composite material. Fig. 7.16 shows the nondimensional axial stress $h\sigma_1/(L\tau_0)$ distribution in the two panels, at two locations across its width $\zeta = 0$ (corresponding to the panel mid-line) and $\zeta = 1$ (corresponding to its upper edge). fig. 7.17 shows the distribution of axial stress at the root of the panel. In contrast with the uniform distribution predicted by classical beam theory, the present results show a significant over-stress occurring at the root corners of the panel, and significant under-stress along its mid-line. The magnitude of the over- and under-stresses are $5 \tanh \mu/\mu$ and $5 \tanh \mu/(2\mu)$, respectively. For the isotropic panel this translates to 26 and 13% for the over- and under-stress, respectively, and the corresponding numbers are 78 and 39% for the composite panel.

Note that the uniform stress distribution results in optimum structural efficiency: the material is equally stressed at all points across the width of the panel. The shear lag phenomenon considerably redistributes the stresses, creating undesirable overstressed area and decreasing structural efficiency. The magnitude of the shear lag effect is controlled by the parameter μ defined in eq. (7.93). The smaller this parameter, the larger the shear lag effect. Clearly, shorter panel (smaller L/h values), made of shear deformable materials (smaller G/E values) will experience the most significant shear lag effects.

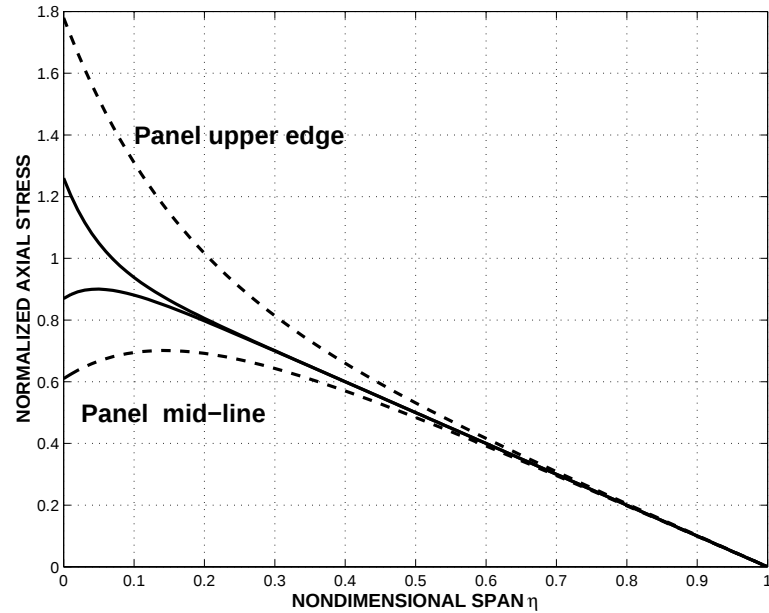


Figure 7.16: The distribution of axial stress along the panel span. Isotropic material: solid line; anisotropic material: dashed line.

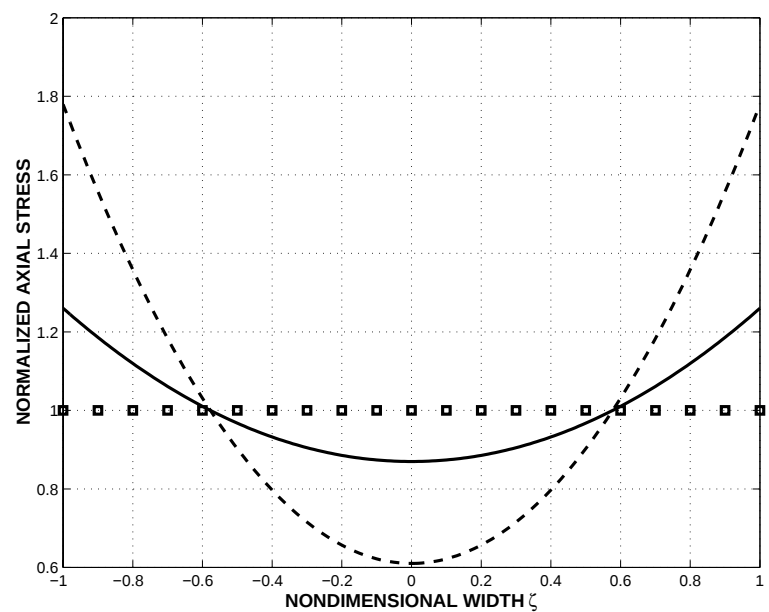


Figure 7.17: The distribution of axial stress across the panel width at the root of the panel. The uniform distribution predicted by beam theory is indicated by the square boxes. Isotropic material: solid line; anisotropic material: dashed line.

Chapter 8

Buckling of Beams

When a structure is subjected to loading it can fail because local stresses exceed the maximum allowable stress for the material. There exist, however, another type of failure mode where the entire structure suddenly collapses. The critical value of the applied load that triggers this failure mode primarily depends on the geometry of the structure and the stiffness of the material, not its strength. The study of this catastrophic failure mode is known as the theory of elastic stability.

The best known problem of elastic stability probably is that of the transverse buckling of a beam. Consider a straight cantilever beam subjected to an end axial compressive load. If this load is applied at the centroid of the cross-section of the beam, it only creates an axial straining of the beam. As the axial compressive load is increased, a critical value is reached where the beam buckles sideways and collapses.

The basic concepts involved in the study of elastic stability will be introduced with the simple problem of a rigid bar with a root torsional spring subjected to a compressive load. This will serve as an introduction to the problem of buckling of beams.

8.1 Rigid bar with root torsional spring

8.1.1 Analysis of a perfect system

Consider a rigid bar of length L articulated at the root as depicted in fig. (8.1). A torsional spring of constant k is acting at the root, and is un-stretched when the bar is vertical. A compressive axial load P constantly acting in the vertical direction is applied at the tip of the bar. Let the lateral deflection of the bar be defined by the angle θ measured from the vertical. The equilibrium of moments about point O implies

$$M - k\theta = 0 \quad (8.1)$$

where M is the moment of the applied load, and $k\theta$ the elastic restoring force in the spring. The moment of the applied load is $M = PL \sin \theta$ resulting in the following equilibrium equation

$$PL \sin \theta - k\theta = 0 \quad (8.2)$$

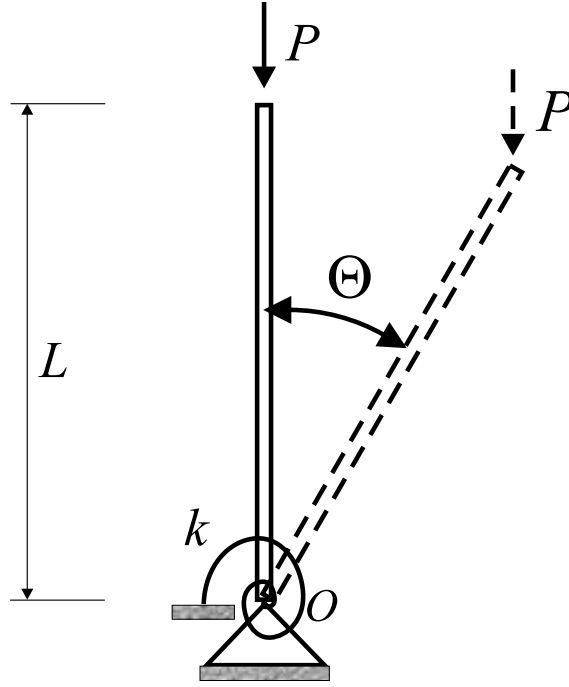


Figure 8.1: Bar with a root torsional spring.

In the theory of elastic stability, the determination of the critical load for the onset of an instability plays a central role. Since the elastic spring is un-stretched when $\theta = 0$, the rigid bar will remain vertical as the applied load P is increased. At the onset of buckling, the bar will start to collapse sideways and θ will increase. If the onset of buckling is to be determined, θ can be assumed to be a small quantity, i.e. one writes

$$\sin(\theta) \approx \theta. \quad (8.3)$$

Hence, (8.2) writes:

$$(PL - k)\theta = 0 \quad (8.4)$$

This equation of equilibrium is satisfied if $\theta = 0$. This represents, however, the trivial solution where the bar remains vertical. Indeed, when the bar is vertical, the line of action of P passes through point O , the moment in the spring vanishes, and equation (8.1) is then identically satisfied. An important point is to determine whether equation (8.4) admits a non-trivial solution. In fact, the critical buckling load of the system P_{cr} is defined as that load for which a non-trivial solution of (8.4) exists. Clearly,

$$P_{cr} = \frac{k}{L} \quad (8.5)$$

When $P = P_{cr}$ equation of equilibrium (8.4) is satisfied for an arbitrary value of θ . The behavior of the system is depicted in fig. (8.2). For $P < P_{cr}$, the only solution of eq. (8.4) is $\theta = 0$. When the applied load reaches the critical buckling load, i.e. $P = P_{cr}$ there exists another solution of eq. (8.4), namely θ is arbitrary.

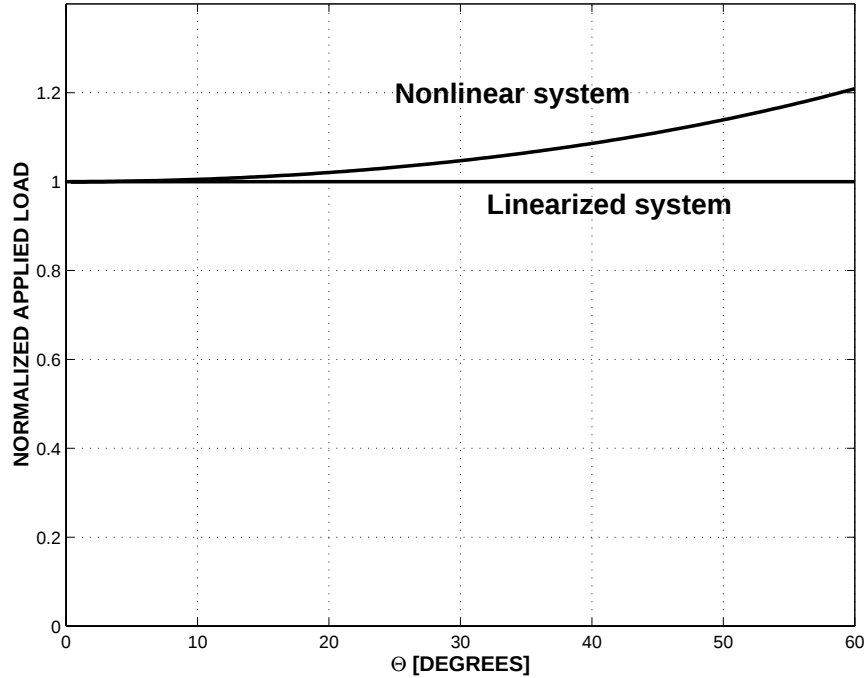


Figure 8.2: Behavior of a perfect system.

Strictly speaking the solution described in fig. (8.2) is only valid for small θ , since assumption (8.3) was made. When θ becomes large, the post-buckling range starts, and eq. (8.2) should be used. This equation is written as

$$\frac{P}{P_{cr}} = \frac{\theta}{\sin \theta} \quad (8.6)$$

This nonlinear relationship is depicted in fig. (8.2). Clearly both solutions are in close agreement for small θ . The critical buckling load characterizes the onset of buckling, i.e. the loading for which significant lateral motion takes place.

From a design standpoint, it is often imperative to keep the applied load well below the critical buckling load as the collapse of the structure under the buckling load is a sudden and catastrophic phenomenon. The buckling load depends on k , the spring stiffness, and L , the length of the bar. The strength of the system components are irrelevant in this analysis.

8.1.2 Analysis of an imperfect system

The system considered above is a *perfect* system in the sense that the rigid bar is perfectly straight, the applied load is perfectly aligned with the bar, and the un-stretched position of the spring corresponds to $\theta = 0$ exactly. In practical situations, however, a certain level of *imperfection* is always present. The actual imperfection of the system is often unknown, as it comes from manufacturing inaccuracies or load misalignment. A convenient way of introducing imperfection in the system is to assume that the un-stretched position of the spring corresponds to $\theta = \theta_0$. θ_0 is now a measure of the initial imperfection of the system.

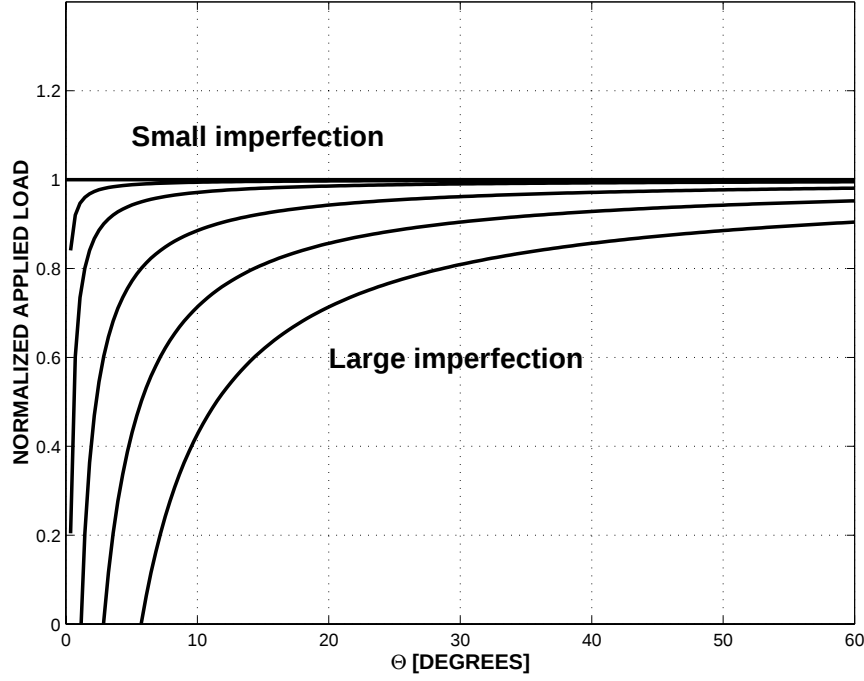


Figure 8.3: Linearized response of the system with various levels of imperfection.

The equilibrium eq. (8.2) then becomes

$$PL \sin \theta - k(\theta - \theta_0) = 0. \quad (8.7)$$

The onset of buckling can be determined assuming θ to be a small quantity, implying (8.3), to find

$$(PL - k)\theta = -k\theta_0. \quad (8.8)$$

This equation possesses a non-zero right hand side in contrast with the homogeneous equation (8.2) found for the perfect system. The solution of eq. (8.8) writes

$$\theta = \frac{\theta_0}{1 - \frac{P}{P_{cr}}} \quad (8.9)$$

The response of the system is depicted in fig. (8.3) for various levels of the initial imperfection θ_0 . When the initial imperfection is very small, the response of the system is very small except when the applied load is very near P_{cr} . When the applied load reaches P_{cr} the response rapidly grows to infinity. For larger initial imperfections, a significant response of the system is observed even for applied loads well below the critical load. In all cases the response of the system is unbounded when the applied load approaches P_{cr} . In fact, the buckling load can be defined as the load for which the response of an initially imperfect system grows without bounds. For the simple system discussed here, both *perfect* and *imperfect* system analyses give the same critical buckling load, eq. (8.5).

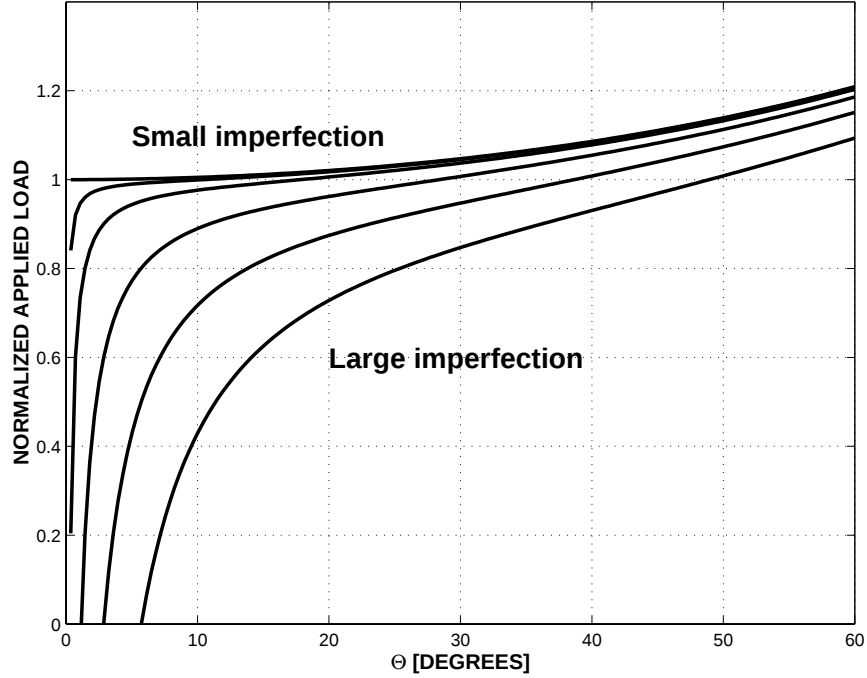


Figure 8.4: Nonlinear response of the system with various levels of imperfection.

The analysis developed here is valid for small values of θ . For larger values of θ equation (8.7) must be used. It can be recast as

$$\frac{P}{P_{cr}} = \frac{\theta - \theta_0}{\sin \theta} \quad (8.10)$$

This relationship is depicted in fig. (8.4) for various levels of the initial imperfection.

Though the critical buckling loads obtained for the *perfect* and *imperfect* systems are the same, their respective behaviors are quite different for applied loads below the critical buckling load. Indeed, comparing figs. (8.2) and (8.3) shows that the *perfect* system presents no lateral deflection for applied loads below P_{cr} , whereas the *imperfect* system presents lateral deflections even for the smallest applied load. In fact, failure can occur for applied loads far smaller than the buckling load. Assuming that the spring fails when $\theta = \theta_{ult}$, the failure load is readily computed from (8.9) as

$$\frac{P_{failure}}{P_{cr}} = 1 - \frac{\theta_0}{\theta_{ult}}. \quad (8.11)$$

It is important to note the fundamental difference between the failure load associated with a displacement or stress reaching an allowable limit for the material ($\theta = \theta_{ult}$ in this simple example), and the critical buckling load associated with an instability of the structure solely dependent on its elastic and geometric characteristics.

In this simple example, two conceptually different definitions of the buckling load were given. First, the buckling load was defined as the critical load for which a *perfect* system admits a non-trivial solution. Second, the buckling load was defined as the load for which

the response of an *imperfect* system grows without bounds. Though conceptually different, these two definitions give the same value of the critical buckling load for the simple rigid bar problem considered here.

8.2 Buckling of beams

Consider now a simply supported, uniform beam acted upon by a compressive load P . At first, this loading is assumed to be applied exactly at the centroid of the section. According to three dimensional beam theory, all the conditions for decoupling the problem are met. This means that three simpler, independent problems can be solved: first, an extensional problem giving rise to an extension U_1 of the beam (compression in this case), and second, two un-coupled bending problems along the principal axes of bending. Since the axial load is applied exactly at the centroid, and no transverse loads are acting, the bending equations are not excited by any loading. As a result, the transverse deflections will remain zero, independently of the level of the applied axial load P .

This simple reasoning clearly shows that the beam theory developed in the previous chapters is unable to predict the buckling phenomenon. The basic equations of Euler-Bernoulli beam theory, eqs. (3.20, and 3.48) must be modified to account for the effect of the large compressive load which creates the instability. The key to this modification is a careful derivation of the equilibrium equations.

8.2.1 Equilibrium equations

Consider an infinitesimal slice of the beam subjected to axial and transverse forces, as well as bending moments. In contrast with earlier developments (see sections (3.4.3) or (3.5.3)), a free body diagram of the *deformed* slice of the beam will be analyzed. Fig. 8.5 shows the deformed slice of the beam subjected to the various force and moment components. At the onset of buckling, a large axial force is present in the beam, but the transverse shear force and deflection are still very small. As a result, the following assumption will be made

$$F_1 \gg F_2; \quad \frac{dU_2}{dx_1} \ll 1. \quad (8.12)$$

The first equilibrium equation is obtained by summing up forces along the $\mathbf{i}_1$ direction to find

$$\begin{aligned} & -F_1 \cos \frac{dU_2}{dx_1} + \left(F_1 + \frac{dF_1}{dx_1} dx_1 \right) \cos \left(\frac{dU_2}{dx_1} + \frac{d^2 U_2}{dx_1^2} dx_1 \right) \\ & + F_2 \sin \frac{dU_2}{dx_1} - \left(F_2 + \frac{dF_2}{dx_1} dx_1 \right) \sin \left(\frac{dU_2}{dx_1} + \frac{d^2 U_2}{dx_1^2} dx_1 \right) + p_1 dx_1 = 0 \end{aligned} \quad (8.13)$$

In view of assumption (8.12) $\cos \frac{dU_2}{dx_1} \approx 1$, and $\sin \frac{dU_2}{dx_1} \approx \frac{dU_2}{dx_1}$. Neglecting higher order terms, eq. (8.13) simplifies to

$$\frac{d}{dx_1} \left(F_1 - F_2 \frac{dU_2}{dx_1} \right) = -p_1.$$

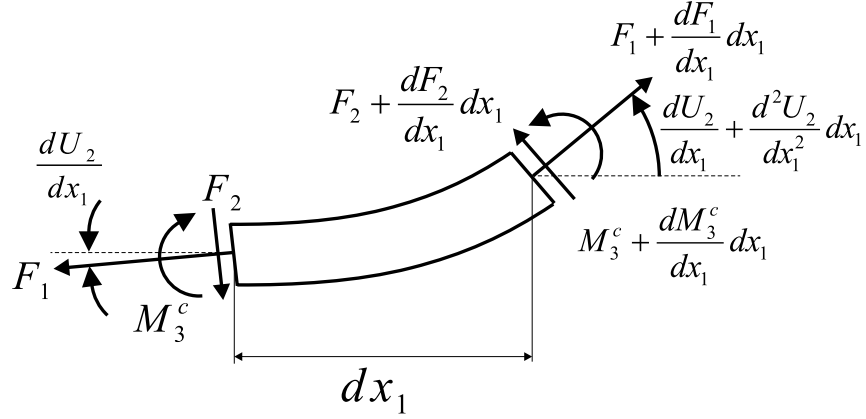


Figure 8.5: Free body diagram of a deformed slice of the beam.

Finally, assumption (8.12) implies that the second term in the parenthesis is negligible as compared to the first, resulting in the following axial equilibrium equation

$$\frac{dF_1}{dx_1} = -p_1. \quad (8.14)$$

This first equation of equilibrium is identical to that obtained earlier (3.20).

The second equation of equilibrium is obtained by summing up the forces along the $\mathbf{i}_2$ direction to find

$$\begin{aligned} & -F_2 \cos \frac{dU_2}{dx_1} + \left(F_2 + \frac{dF_2}{dx_1} dx_1 \right) \cos \left(\frac{dU_2}{dx_1} + \frac{d^2U_2}{dx_1^2} dx_1 \right) \\ & -F_1 \sin \frac{dU_2}{dx_1} + \left(F_1 + \frac{dF_1}{dx_1} dx_1 \right) \sin \left(\frac{dU_2}{dx_1} + \frac{d^2U_2}{dx_1^2} dx_1 \right) + p_2 dx_1 = 0 \end{aligned} \quad (8.15)$$

Here again, this equation considerably simplifies when assumptions (8.12) are taken into account; it becomes

$$\frac{d}{dx_1} \left(F_2 + F_1 \frac{dU_2}{dx_1} \right) = -p_2.$$

The two terms in the parenthesis are now of the same order of magnitude. The final equilibrium equation cannot be further simplified and writes

$$\frac{dF_2}{dx_1} + \frac{d}{dx_1} \left(F_1 \frac{dU_2}{dx_1} \right) = -p_2 \quad (8.16)$$

This equilibrium equation differs from that derived earlier (3.46). The second term represents the contribution of the large axial force F_1 to the transverse equilibrium of the beam.

The last equilibrium equation is obtained by summing up the moments about $\mathbf{i}_3$ to find

$$-M_3^c + \left(M_3^c + \frac{dM_3^c}{dx_1} dx_1 \right) + F_2 dx_1 = 0. \quad (8.17)$$

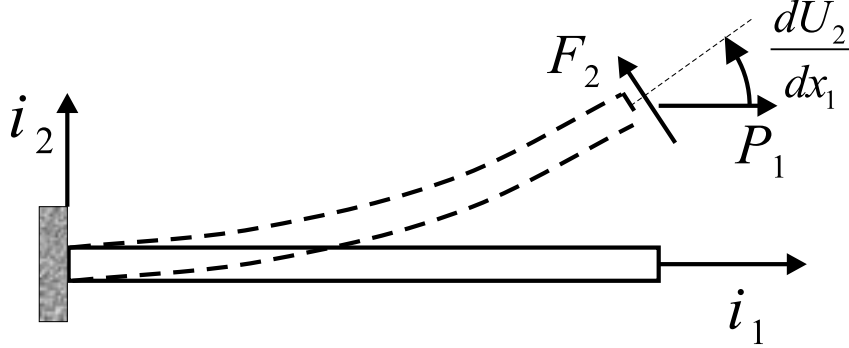


Figure 8.6: Cantilevered beam with a tip load.

After simplification this equation writes

$$\frac{dM_3^c}{dx_1} + F_2 = 0. \quad (8.18)$$

Once again this equation is identical to that obtained earlier (3.46).

The governing equation is obtained by eliminating the shear force F_2 from eqs. (8.16) and (8.18), then introducing the bending moment, eq. (3.44) to find

$$\frac{d^2}{dx_1^2} \left[I_{33}^c \frac{d^2 U_2}{dx_1^2} \right] - \frac{d}{dx_1} \left[F_1 \frac{dU_2}{dx_1} \right] = p_2. \quad (8.19)$$

The second term in this equation is a new term which was absent in previous developments, see governing equation (3.48). The governing equation is now a fourth order differential equation for the transverse displacement U_2 . Four boundary conditions are required, two at each end of the beam. The boundary conditions are identical to those discussed in section (3.5.4), except when a large axial force is applied at one end of the beam. In that case, the boundary conditions must be derived from considering equilibrium of the deformed configuration of the beam. Consider for instance, a cantilevered beam with an end tip load P_1 acting in a fixed, horizontal direction, as depicted in fig. 8.6. The boundary conditions at the tip end are:

$$M_3^c = 0; \quad F_2 = -P_1 \frac{dU_2}{dx_1}. \quad (8.20)$$

Governing equation (8.19) can be applied when large axial forces are present in the beam, whether in compression (F_1 is negative) or in tension (F_1 is positive). If the axial force is compressive, the beam will buckle when this compressive force reaches a critical level; if the axial force is tensile, the beam will not buckle, though its behavior will be affected by the presence of the large axial force.

8.2.2 Buckling of a pinned-pinned beam (Equilibrium approach)

Consider a uniform, pinned-pinned beam subjected to end compressive loads of magnitude P . The resulting axial force F_1 in the beam is constant along the span, and $F_1 = -P$.

Assuming the axis $\mathbf{i}_2$ to be a principal axis of bending, the un-coupled governing equation that accounts for the presence of the large compressive load writes

$$I_{33}^{*c} \frac{d^4 U_2}{dx_1^4} + P \frac{d^2 U_2}{dx_1^2} = 0. \quad (8.21)$$

with the following boundary conditions

$$@x_1 = 0 : U_2 = \frac{d^2 U_2}{dx_1^2} = 0; \quad @x_1 = L : U_2 = \frac{d^2 U_2}{dx_1^2} = 0. \quad (8.22)$$

The governing equations and associated boundary conditions are homogeneous. Hence, the trivial solution $U_2 \equiv 0$ is a solution of the problem. The critical buckling load is defined as the lowest load for which a non-trivial solution of the governing equations exists.

For simplicity the governing equation is rewritten as

$$U_2'''' + \lambda^2 U_2'' = 0, \quad (8.23)$$

where

$$\lambda^2 = \frac{PL^2}{I_{33}^{*c}} \quad (8.24)$$

is a non-dimensional loading parameter, and $(.)'$ denotes a derivative with respect to the non-dimensional span variable $\xi = x_1/L$. The boundary conditions become

$$@\xi = 0 : U_2 = U_2'' = 0; \quad @\xi = L : U_2 = U_2'' = 0. \quad (8.25)$$

The solution of eq. (8.23) is

$$U_2 = A + B\xi + C \cos \lambda\xi + D \sin \lambda\xi \quad (8.26)$$

where A , B , C , and D , are four integration constants to be determined from the boundary conditions (8.25). The first two boundary conditions imply

$$0 = A + C; \quad 0 = \lambda^2 C,$$

which in turn imply $A = C = 0$. The last two boundary conditions yield

$$\left[\begin{array}{cc} L & \sin \lambda \\ 0 & \sin \lambda \end{array} \right] \left| \begin{array}{c} B \\ D \end{array} \right| = 0. \quad (8.27)$$

This is a set of algebraic equations for the last two integration constants B and D . Since the system is homogeneous, the solution is $B = D = 0$, which corresponds to the trivial solution of the problem. A linear system of homogeneous algebraic equations admits a non-trivial solution if and only if the determinant of the system is zero, i.e. when

$$\det \left[\begin{array}{cc} L & \sin \lambda \\ 0 & \sin \lambda \end{array} \right] = 0. \quad (8.28)$$

Expanding this determinant yield the buckling equation

$$\sin \lambda = 0. \quad (8.29)$$

The governing equation of the problem admits non-trivial solutions for the values λ_n which satisfy this equation, and the lowest solution is the critical buckling load. The roots of (8.29) are $\lambda_n = n\pi$, $n = 1, 2, \dots, \infty$, which, in view of (8.24), write

$$(P_{cr})_n = \frac{n^2 \pi^2 I_{33}^{*c}}{L^2}; \quad n = 1, 2, \dots, \infty. \quad (8.30)$$

The lowest root corresponds to $n = 1$, resulting in a critical buckling load

$$P_{cr} = \frac{\pi^2 I_{33}^{*c}}{L^2}. \quad (8.31)$$

The buckling mode shape can be readily determined from this analysis. The first equation in (8.27) is $BL + D \sin \lambda = 0$, which, in view of (8.29), yields $B = 0$. The buckling mode shape is then $U_2 = D \sin \lambda_n \xi$. The mode shape corresponding to the lowest buckling load is now

$$U_2 = D \sin \pi \xi. \quad (8.32)$$

As the beam buckles, its transverse deflection is of sinusoidal shape. The integration constant D remains undetermined, or, in other words, the amplitude of this transverse deflection is of arbitrary amplitude, indicating a transverse collapse of the beam. A more precise description of the behavior of the structure at buckling cannot be determined within the framework of this linearized theory.

The above analysis is based on the uncoupled governing equation (8.21) pertaining to bending in the $\mathbf{i}_1 \mathbf{i}_2$ plane. A buckling analysis using the uncoupled governing equation in the $\mathbf{i}_1 \mathbf{i}_3$ plane could be performed in the exact same manner, leading to the following buckling load

$$P_{cr} = \frac{\pi^2 I_{22}^{*c}}{L^2}. \quad (8.33)$$

From a practical standpoint, the critical buckling load is the lowest of (8.31) and (8.33), where I_{33}^{*c} and I_{22}^{*c} are the principal centroidal bending stiffnesses. The critical buckling load, called the *Euler buckling load* is

$$P_{Euler} = \frac{\pi^2 I^{*c}}{L^2}, \quad (8.34)$$

where I^{*c} is the lowest principal centroidal bending stiffness. This buckling load is proportional to this lowest principal centroidal bending stiffness, and inversely proportional to the square of the beam length.

8.2.3 Buckling of a pinned-pinned beam (Imperfection approach)

In the previous section a *perfect* system was analyzed. The beam is perfectly uniform and straight, and the load is exactly applied at the centroid. In practical situations, no system is ever perfect: manufacturing imperfections always result in non-uniform beams, and no

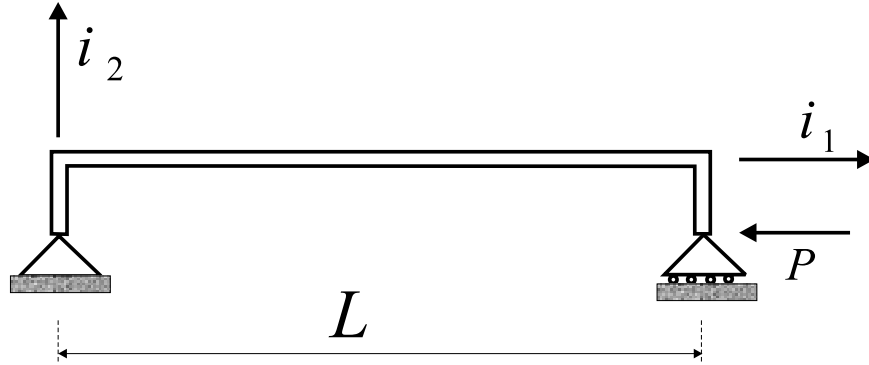


Figure 8.7: Pinned-pinned beam with excentric end loads.

experimental set-up can apply a compressive load exactly at the centroid. Such imperfections are always present, though their exact nature and magnitude are generally unknown.

To investigate the effect of these imperfections, a pinned-pinned beam with eccentrically applied end compressive loads will be analyzed. Fig. 8.7 depicts the geometry of the problem: the end compressive loads are applied at a distance e from the centroid. The governing equation of the problem is (8.21) once more. However, the boundary conditions are now

$$@x_1 = 0 : U_2 = 0, \quad I_{33}^* \frac{d^2 U_2}{dx_1^2} = Pe; \quad @x_1 = L : U_2 = 0, \quad I_{33}^* \frac{d^2 U_2}{dx_1^2} = Pe. \quad (8.35)$$

The problem defined by (8.21) and (8.35) is no longer a homogeneous problem, due to the non-zero right hand side appearing in the boundary conditions. The trivial solution $U_2(x_1) \equiv 0$ is not a solution of the problem. Introducing once more the non-dimensional span and loading variables ξ and λ , respectively, the differential equation can be written as (8.23). The boundary conditions now become:

$$@\xi = 0 : U_2 = 0, \quad U_2'' = -\lambda^2 e; \quad @\xi = L : U_2 = 0, \quad U_2'' = -\lambda^2 e. \quad (8.36)$$

The solution of (8.23) is once more given by (8.26). The boundary conditions at $\xi = 0$ imply

$$A + C = 0; \quad -\lambda^2 C = -\lambda^2 e;$$

which yields $-A = C = e$. Proceeding with the boundary conditions at $\xi = 1$ yields

$$B + D \sin \lambda = e(1 - \cos \lambda); \quad -\lambda^2 D \sin \lambda = -e\lambda^2(1 - \cos \lambda).$$

Solving for the last two integration constants gives $D = e(1 - \cos \lambda)/\sin \lambda$ and $B = 0$. The final solution for the transverse displacement writes

$$U_2 = e \left[\frac{1 - \cos \lambda}{\sin \lambda} \sin \lambda \xi + \cos \lambda \xi - 1 \right] = e \left[\frac{\cos \lambda (\xi - \frac{1}{2})}{\cos \frac{\lambda}{2}} - 1 \right] \quad (8.37)$$

The buckling load of the structure can be defined as the load for which the response of an initially imperfect structure ceases to be bounded. Clearly, $U_2 \rightarrow \infty$ when $\cos \frac{\lambda}{2} \rightarrow 0$, i.e. when $\lambda = (2n - 1)\pi$, $n = 1, 2, \dots, \infty$. For (8.24), the dimensional buckling load now writes

$$(P_{cr})_n = \frac{(2n - 1)^2 \pi^2 I_{33}^{*c}}{L^2}; \quad n = 1, 2, \dots, \infty. \quad (8.38)$$

The lowest buckling load corresponds to $n = 1$, leading to (8.31) once again. The equilibrium and imperfection approaches give the same critical buckling load for this specific case.

Insight into the behavior of imperfect structures can be gained by computing the mid-span deflection δ of the beam. With $\xi = \frac{1}{2}$, (8.37) becomes

$$\delta = U_2(\xi = \frac{1}{2}) = e \left[\frac{1}{\cos \frac{\lambda}{2}} - 1 \right]$$

The non-dimensional loading parameter is now expressed in terms of the applied compressive load with the help of (8.24) and (8.34)

$$\lambda = \left[\frac{PL^2}{I^{*c}} \right]^{1/2} = \left[\frac{\pi^2 P}{P_{Euler}} \right]^{1/2} = \pi \left[\frac{P}{P_{Euler}} \right]^{1/2}. \quad (8.39)$$

The mid-span transverse deflection now writes

$$\delta = e \left[\frac{1}{\cos \left(\frac{\pi}{2} \left[\frac{P}{P_{Euler}} \right]^{1/2} \right)} - 1 \right] \quad (8.40)$$

Fig. 8.8 depicts the mid-span deflection as a function of the applied compressive load P , for various levels of the initial imperfection e . For all levels of imperfection, the mid-span deflection grows to infinity as the applied load approaches the critical buckling load. Though large mid-span deflections can occur for applied loads far smaller than the critical buckling load when large imperfections are present, the critical buckling load itself is unaffected by the presence of imperfections.

The bending moment distribution in the beam is readily computed from the transverse deflection distribution eq. (8.37) as

$$M_3^c = Pe \frac{\cos \lambda(\xi - \frac{1}{2})}{\cos \frac{\lambda}{2}}. \quad (8.41)$$

The maximum bending moment occurs in the middle of the beam and is

$$M_{MAX} = \frac{Pe}{\cos \frac{\lambda}{2}} \quad (8.42)$$

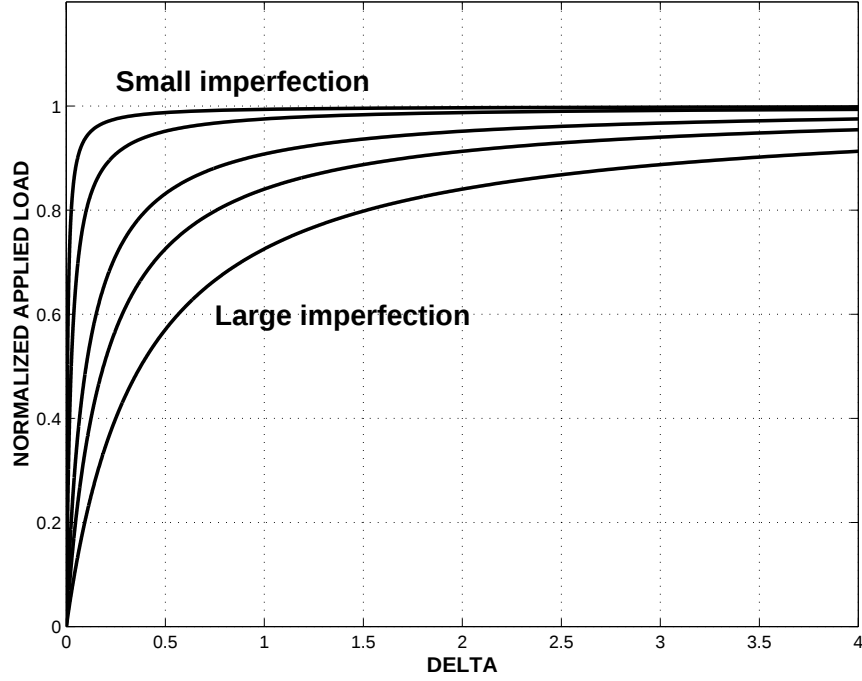


Figure 8.8: Mid-Span transverse deflection of an imperfect pinned-pinned beam for various levels of imperfection.

Considering a beam with a rectangular cross-section of height h made of a homogeneous material with a modulus of elasticity E , the corresponding maximum axial stress in the beam becomes

$$\sigma_{MAX} = \frac{\pi^2}{2} \left(\frac{he}{L^2} \right) \frac{P/P_{Euler}}{E \cos \frac{\pi}{2} [P/P_{Euler}]^{1/2}}. \quad (8.43)$$

The beam will fail when the maximum axial stress reaches the ultimate allowable stress σ_{ult} , i.e. when

$$\frac{P_{failure}/P_{Euler}}{\cos \frac{\pi}{2} [P_{failure}/P_{Euler}]^{1/2}} = \frac{2}{\pi^2} \left(\frac{\sigma_{ult}}{E} \right) \left(\frac{L^2}{he} \right) \quad (8.44)$$

For structures with large imperfections, the failure load $P_{failure}$ can be substantially lower than the buckling load P_{Euler} , as depicted in fig. 8.9.

8.2.4 Buckling of beams with various end conditions.

The analysis method developed in the previous section can be repeated for beams with various end conditions. The critical buckling load for various configurations are summarized in table 8.1 which lists the non-dimensional buckling parameter k , such that

$$P_{cr} = k \frac{I^{*c}}{L^2}.$$

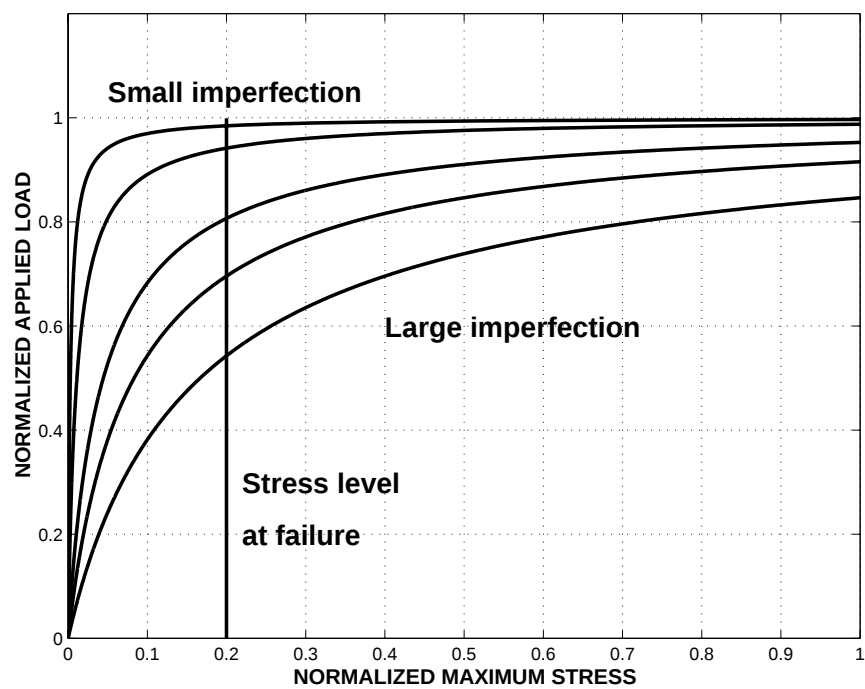


Figure 8.9: Maximum stress in the beam versus applied compressive load.

Boundary Conditions	Buckling parameter k
Clamped - Free	$\pi^2/4$
Clamped - Clamped	$4\pi^2$
Pinned - Pinned	π^2

Table 8.1: Buckling loads for beams with various end conditions.

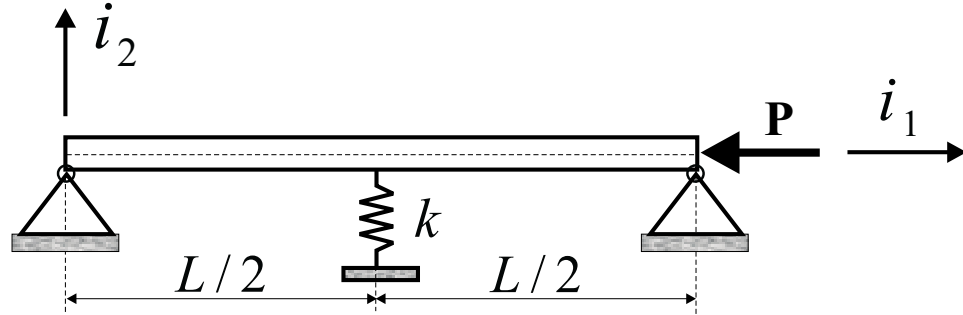


Figure 8.10: Simply supported beam with a mid-span spring.

8.2.5 Problems

Problem 8.1

Study the effect of a midspan spring of stiffness constant k on the critical buckling load of the simply supported beam of span L depicted in fig. 8.10. To investigate this problem, use an energy approach with the following assumed mode shapes

$$U_2(\eta) = a_1 \sin \pi \eta + a_2 \sin 2\pi \eta + a_3 \sin 3\pi \eta.$$

It is convenient to use the following notation: $P_e = \pi^2 I_{33}^c / L^2$ is the Euler buckling load for the beam in the absence of the mid-span spring, $k^* = kL^3 / I_{33}^c$ the nondimensional stiffness of the spring, and $\eta = x_1 / L$ the nondimensional span variable.

1. Find the buckling loads of the system.
2. How does the lowest buckling load vary when k^* increases? Plot the nondimensional buckling loads P/P_e as a function of k^* .
3. How much improvement in buckling load can be expected from the mid-span spring.

Problem 8.2

A uniform cantilevered beam of span L has a tip support, as depicted in fig. 8.11. The beam is subjected to a tip compressive axial force P and a uniform transverse loading p_0 .

1. Find the exact solution of the problem when the beam is *subjected to the sole transverse load* p_0 .
2. Find an approximate solution for the problem when the beam is *subjected to the combined loading*, *i.e.* both transverse loading p_0 and tip compressive load P are applied. Use an energy method with a single assumed mode selected to be the solution to part 1.
3. Determine the buckling load P_{cr} for this problem.
4. How is the buckling load affected by the transverse loading p_0 .

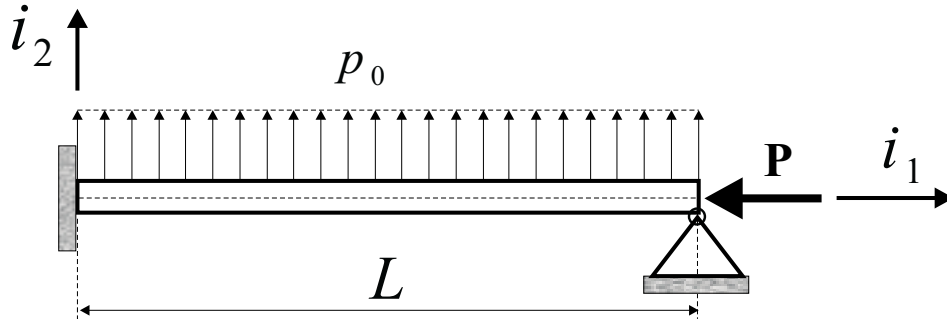


Figure 8.11: Cantilevered beam with tip support subjected to a compressive load.

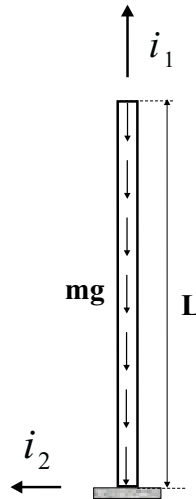


Figure 8.12: Flag pole standing under its own weight.

5. Under the combined loading condition, find the failure envelope in the two-dimensional space P/P_{cr} , $p_0/(b\sigma_{ult})$. Failure is reached when $|\sigma_{ben}^{\max}| + |\sigma_{axi}^{\max}| = \sigma_{ult}$, where $\sigma_{ben}^{\max}$ and $\sigma_{axi}^{\max}$ are the the maximum stresses due to bending and axial force, respectively, and σ_{ult} the ultimate allowable stress for the material. Use the following data $E = 73.0\text{GPa}$, $\sigma_{ult} = 620\text{MPa}$. Plot two failure envelopes for $L/h = 20$ and 10 on the same graph. Assume a rectangular section of width b and height h .

Problem 8.3

A flagpole of uniform mass per unit span m is standing up against gravity, as depicted in fig. 8.12. What is the critical weight mg_{cr} such that this flagpole will buckle under its own weight?

1. Use an energy method to solve this problem. Use the following assumed mode

$$U_2(\eta) = a (1 - \cos \pi\eta/2).$$

2. If the flagpole is made of steel and has a square cross-section ($1\text{cm} \times 1\text{cm}$), what is the

critical length at which it will buckle under its own weight. (For steel: material density $\rho = 7700\text{kg/m}^3$, Young's modulus $E = 210\text{GPa}$; $g = 9.81\text{m/sec}^2$).

Chapter 9

Vibration of Beams

In the previous chapters, the deflection of beams under static loading was investigated. Static loading means that the external forces acting on the beam were applied so slowly that the inertial forces associated with the motion of the beam can be neglected. In numerous applications, inertial forces are significant and drastically alter the response of the beam. For instance, if a beam is vibrating after being struck by an impulsive force, inertial forces are the only forces acting on the beam.

The various aspects of this complex dynamic problem will be introduced by considering first a far simpler problem, namely that of a simple spring-mass system.

9.1 Analysis of a spring-mass system

Fig. 9.1 depicts a simple spring mass system and the associated free body diagram which yields the following dynamic equation of equilibrium

$$F(t) - kx - m\ddot{x} = 0, \quad (9.1)$$

where t is time and $(\dot{})$ denotes a time derivative; m is the mass and $m\ddot{x}$ the inertial force; k the spring constant and kx the elastic restoring force; and $F(t)$ the time dependent externally applied force. This externally applied force is assumed to be sinusoidal, with an amplitude F_0 and a frequency Ω . Eq. (9.1) is now rewritten as

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m} \sin \Omega t. \quad (9.2)$$

This is the equation of motion of the system which is in the form of a second order, ordinary differential equation in time. The excitation force appears on the right hand side of the equation.

9.1.1 The homogeneous problem

At first, we focus on the homogeneous problem which represents the behavior of the system in the absence of externally applied forces. The governing equation

$$\ddot{x} + \frac{k}{m}x = 0, \quad (9.3)$$

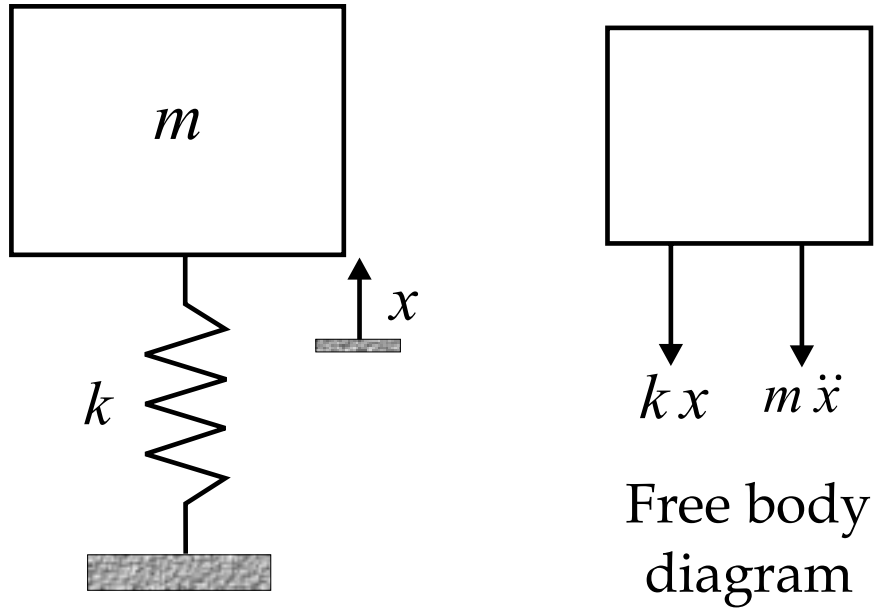


Figure 9.1: Simple spring mass system and free body diagram.

clearly admits the trivial solution $x(t) \equiv 0$. However, this solution is not very useful. An important question is whether eq. (9.3) admits a *non-trivial solution*. Consider a solution of the form

$$x(t) = X_0 e^{i\omega t}, \quad (9.4)$$

where $i = \sqrt{-1}$. This represents a harmonic motion of amplitude X_0 , at an unknown frequency ω . Introducing this assumed motion in eq. (9.3) yields

$$\left(-\omega^2 + \frac{k}{m}\right) X_0 e^{i\omega t} = 0. \quad (9.5)$$

Since $e^{i\omega t} \neq 0$, we obtain

$$\left(-\omega^2 + \frac{k}{m}\right) X_0 = 0. \quad (9.6)$$

The time dependency has now disappeared from the equation; it means that the assumed harmonic time dependency (9.4) is indeed the solution of the homogeneous equation of motion (9.3). The algebraic equation (9.6) admits two solutions

$$X_0 = 0; \quad \omega^2 = \frac{k}{m}. \quad (9.7)$$

Introducing the first solution into eq. (9.4) yields $x(t) \equiv 0$, the trivial solution of the problem. The second solution $\omega = \sqrt{k/m}$ gives the specific frequency of the harmonic motion (9.4) which satisfies the homogeneous differential equation of motion (9.3). This frequency is called the *natural frequency* of the system. Harmonic motion at the natural frequency is a non-trivial solution of the homogeneous equation of motion of the system.

9.1.2 The forced problem

The significance of the natural frequency is best understood when analyzing the response of the system under an externally applied force. In that case, equation of motion (9.2) writes

$$\ddot{x} + \omega^2 x = \omega^2 X_s \sin \Omega t. \quad (9.8)$$

where $X_s = F_0/k$ is the static response of the system, i.e. the deflection of the spring-mass system if the force F_0 were applied so slowly that the inertial force were negligible. The solution of equation (9.8) consists of two parts: the solution of the homogeneous equation, and a particular solution for the non-zero right hand side. The complete solution is

$$x(t) = A \cos \omega t + B \sin \omega t + \frac{\omega^2 X_s}{\omega^2 - \Omega^2} \sin \Omega t \quad (9.9)$$

where A and B are integration constants to be determined from the initial conditions. If the system is initially at rest, i.e. $x(t=0) = 0$ and $\dot{x}(t=0) = 0$, the integration constants can be determined to yield the response of the system

$$\frac{x(t)}{X_s} = \frac{\sin \Omega t}{1 - \left(\frac{\Omega}{\omega}\right)^2} - \left(\frac{\Omega}{\omega}\right) \frac{\sin \omega t}{1 - \left(\frac{\Omega}{\omega}\right)^2}. \quad (9.10)$$

The first term is the **forced response** of the system, i.e. the response of the system at the frequency Ω of the excitation. The second term represents a response of the system at its own natural frequency ω . Clearly the natural frequency of the system plays an important role since part of the response of the system is at its natural frequency, no matter what the excitation frequency is. Furthermore, the natural frequency also appears in the forced response of the system. The norm of the amplitude of the forced response is:

$$\left| \frac{x(t)}{X_s} \right| = \left| \frac{1}{1 - \left(\frac{\Omega}{\omega}\right)^2} \right|. \quad (9.11)$$

Fig. 9.2 depicts the amplitude of the forced response as a function of the excitation frequency. The important features of this response can be described as follows. For $\Omega/\omega \ll 1$, i.e. at excitation frequencies far below the natural frequency, the response of the system is identical to its static response. For $\Omega/\omega \gg 1$, i.e. at excitation frequencies far greater than the natural frequency, the response of the system tends to zero. Indeed, the inertial force is proportional to the acceleration of the mass, itself proportional to Ω^2 . At high excitation frequencies, high inertial forces are generated which oppose the motion, resulting in a vanishing system response. For $\Omega \approx \omega$, the response of the system becomes very large. In fact, when the excitation frequency is equal to the natural frequency, the response of the system grows to infinity. The excitation frequency is said to be **in resonance** with the natural frequency of the system.

The above discussion shows that the knowledge of the natural frequency is key to the analysis of the response of the system.

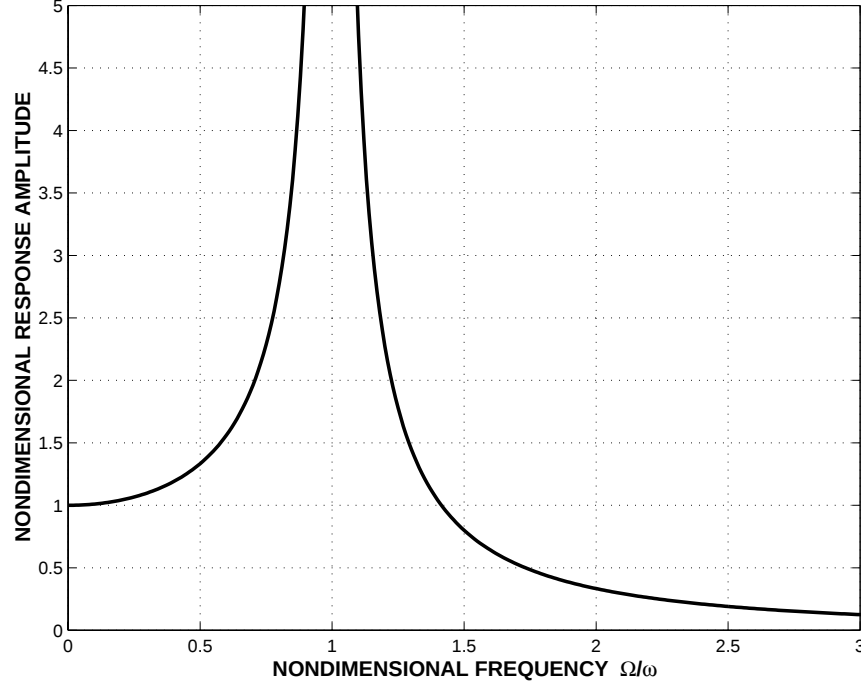


Figure 9.2: Forced response of spring mass system.

The prediction of the dynamic response of elastic structures is a far more complex task than that of the dynamic response of a simple spring-mass system. Yet, the main features characterizing the response of the simple spring-mass system are found in the response of elastic structures. There is, however, a fundamental difference: elastic structures possess an infinite number of natural frequencies as opposed to the single natural frequency present in the spring-mass system. This will be demonstrated in the following section dealing with a simply supported beam.

9.2 Analysis of a simply supported beam

Consider a uniform beam of length L simply supported at both ends. The governing equation is given by (3.48)

$$\frac{d^2}{dx_1^2} \left[I_{33}^c \frac{d^2 U_2}{dx_1^2} \right] = p_2 \quad (9.12)$$

with the following boundary conditions:

$$@x_1 = 0 : \quad U_2 = 0; \quad \frac{d^2 U_2}{dx_1^2} = 0; \quad @x_1 = L : \quad U_2 = 0; \quad \frac{d^2 U_2}{dx_1^2} = 0. \quad (9.13)$$

When the beam is vibrating, it is subjected to inertial forces which are equivalent to externally applied distributed transverse loads of the form

$$p_2(x_1) = -m \frac{d^2 U_2}{dt^2}, \quad (9.14)$$

where t is time, $\frac{d^2 U_2}{dt^2}$ the transverse acceleration, and m the mass of the beam per unit span. In the absence of other applied loads and assuming the beam to be uniform, the governing eq. (3.48) becomes

$$I_{33}^c \frac{d^4 U_2}{dx_1^4} + m \frac{d^2 U_2}{dt^2} = 0. \quad (9.15)$$

This is the equation of motion of the problem, a partial differential equation for the transverse deflection $U_2(x_1, t)$. It is fourth order in the spatial variable x_1 , and second order in time. At all times, it is subjected to the boundary conditions (9.13). The governing equation is homogeneous, as are the boundary conditions. Hence, the trivial solution $U_2(x_1, t) \equiv 0$ clearly is a solution. This is easily understood in physical terms. If the beam is not moving, i.e. $U_2(x_1, t) = 0$, no inertial forces are generated, and no loads are applied on the beam. The response of the system is then clearly $U_2(x_1, t) = 0$. The goal of this analysis is to determine when eq. (9.15) admits a non trivial solution.

As was done in the case of the spring-mass system, the time dependency of the motion is assumed to be harmonic

$$U_2(x_1, t) = \bar{U}_2(x_1) e^{i\omega t} \quad (9.16)$$

where $\bar{U}_2(x_1)$ is a function of the spatial variable x_1 only, $e^{i\omega t}$ the assumed harmonic time dependency of the solution, and ω its frequency. The validity of the assumed solution is readily verified by introducing (9.16) into (9.15) to find

$$\left(I_{33}^c \frac{d^4 \bar{U}_2}{dx_1^4} - m\omega^2 \bar{U}_2 \right) e^{i\omega t} = 0.$$

Since $e^{i\omega t} \neq 0$, the following ordinary differential equation is obtained

$$I_{33}^c \frac{d^4 \bar{U}_2}{dx_1^4} - m\omega^2 \bar{U}_2 = 0. \quad (9.17)$$

Similarly, introducing (9.16) in the boundary conditions (9.13) yields

$$@x_1 = 0 : \quad \bar{U}_2 = 0; \quad \frac{d^2 \bar{U}_2}{dx_1^2} = 0; \quad @x_1 = L : \quad \bar{U}_2 = 0; \quad \frac{d^2 \bar{U}_2}{dx_1^2} = 0. \quad (9.18)$$

The time dependency has now disappeared from the governing equation and associated boundary conditions, verifying that the harmonic time dependency (9.16) is indeed the time dependency of the solution.

The solution of (9.17) is eased by introducing the non-dimensional span variable $\xi = x_1/L$. A change of variable in eq. (9.17) yields

$$\bar{U}_2'''' - \bar{\omega}^2 \bar{U}_2 = 0. \quad (9.19)$$

subjected to:

$$@\xi = 0 : \quad \bar{U}_2 = 0; \quad \bar{U}_2'' = 0; \quad @\xi = 1 : \quad \bar{U}_2 = 0; \quad \bar{U}_2'' = 0. \quad (9.20)$$

where the notation $(.)'$ was used for $\frac{d}{d\xi}$, and the non-dimensional frequency $\bar{\omega}$ was defined as

$$\bar{\omega} = \omega \left[\frac{mL^4}{I_{33}^c} \right]^{1/2}. \quad (9.21)$$

The problem has now been reduced to the ordinary differential equation (9.19) with the associated boundary conditions (9.20). This contrasts with the original problem (9.15) which was in the form of a partial differential equation. Clearly, the trivial solution $\bar{U}_2(\xi) \equiv 0$ is still a solution of the problem. The frequency ω of the harmonic motion is as yet undetermined as the harmonic time dependency (9.16) satisfies the governing equations for any ω .

The solution of (9.19) is of the form

$$\bar{U}_2 = A \cos k\xi + B \sin k\xi + C \cosh k\xi + D \sinh k\xi, \quad (9.22)$$

where $k = \sqrt{\bar{\omega}}$, and A , B , C , and D integration constants to be determined from the boundary condition (9.20). Enforcing the first two boundary condition at $\xi = 0$ yields $A = C = 0$. Proceeding with the two boundary conditions at $\xi = 1$, the following algebraic equations are found for the two last integration constants B and D

$$\begin{bmatrix} \sinh k & \sin k \\ \sinh k & -\sin k \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = 0. \quad (9.23)$$

This system of algebraic equations is homogeneous, resulting in the solution $B = D = 0$. However, since $A = C = 0$, this corresponds to the trivial solution $\bar{U}_2(\xi) = 0$. A set of homogeneous algebraic equations admits a non-trivial solution if and only if the determinant of the system vanishes. In other words, system (9.23) admits a non-trivial solution if and only if

$$\det \begin{bmatrix} \sinh k & \sin k \\ \sinh k & -\sin k \end{bmatrix} = 0. \quad (9.24)$$

Expanding this determinant yields the *frequency equation*:

$$\sinh k \sin k = 0 \quad (9.25)$$

The values of k (or equivalently of $\bar{\omega}$) that satisfy this equation give rise to a non-trivial solution of the problem. $\sinh k = 0$ yields a single solution $k = 0$ which implies $\bar{U}_2(\xi) = 0$ once again; $\sin k = 0$ yields an infinite number of solutions denoted $k_n = n\pi$, $n = 1, 2, \dots, \infty$. In terms of the non-dimensional frequency this writes $\bar{\omega}^2 = n^2\pi^2$, and finally the corresponding dimensional frequencies are found with the help of (9.21)

$$\omega_n = n^2\pi^2 \left[\frac{I_{33}^c}{mL^4} \right]^{1/2}, \quad n = 1, 2, \dots, \infty. \quad (9.26)$$

The mode shapes corresponding to these natural frequencies are found by introducing the solution for k_n into the first equation of system (9.23) to find $B_n \sinh n\pi + D_n \sinh n\pi = 0$. Since $\sin n\pi = 0$ this yields $B_n = 0$, and the mode shapes are then

$$\bar{U}_{2n}(\xi) = D_n \sin n\pi\xi, \quad n = 1, 2, \dots, \infty. \quad (9.27)$$

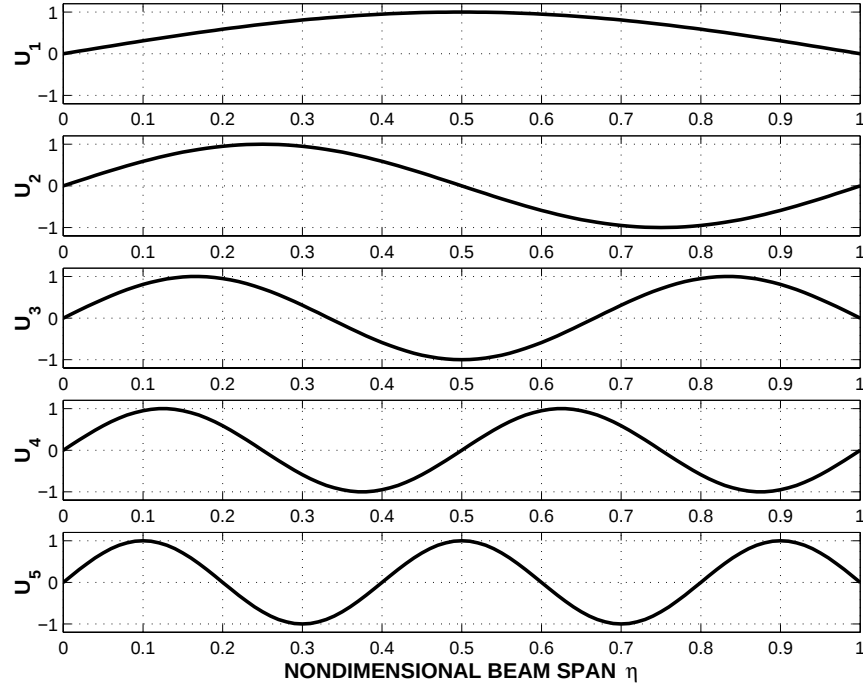


Figure 9.3: Five lowest eigen-modes of a simply supported beam.

The amplitude D_n of these mode shapes remains arbitrary, this means that the mode shape is a non-trivial solution of the problem for any arbitrary amplitude D_n . The first few lowest modes are depicted in fig. 9.3. The terms *eigen frequencies* and *eigen modes* are often used instead of *natural frequencies* and *natural mode shapes*, respectively.

The simply supported beam possesses a spectrum of natural frequencies defined by (9.26). If the beam is acted upon by an external harmonic force with an excitation frequency Ω equal to any of these natural frequencies large response amplitude will result.

9.3 Uniform beams with various end conditions

The analysis method developed in the previous section can be repeated for beams with various end conditions. The procedure is identical up to the determination of the integration constants appearing in the general solution (9.22). The spectrum of natural frequencies for the various end conditions is summarized in table 9.1 which lists the non-dimensional parameters k_n , such that

$$\omega_n = k_n^2 \left[\frac{I_{33}^c}{mL^4} \right]^{1/2}, \quad n = 1, 2, \dots \infty.$$

Boundary Conditions	k_1	k_2	k_3	$k_n, \quad n = 4, 5, \dots \infty$
Free - Free	0	4.730	7.853	$\approx (2n - 1) \pi/2$
Free - Guided	0	2.365	5.498	$\approx (4n - 5) \pi/2$
Free - Pinned	0	3.927	7.069	$\approx (4n - 3) \pi/4$
Guided - Guided	0	3.142	6.283	$= (n - 1) \pi$
Guided - Pinned	1.561	5.712	7.854	$= (2n - 1) \pi/2$
Clamped - Free	1.875	4.694	7.855	$\approx (2n - 1) \pi/2$
Pinned - Pinned	3.142	6.283	9.425	$= n\pi$
Clamped - Pinned	3.927	7.069	10.210	$\approx (4n + 1) \pi/4$
Clamped - Guided	2.365	5.498	8.639	$\approx (4n - 1) \pi/4$
Clamped - Clamped	4.730	7.853	10.996	$\approx (2n + 1) \pi/2$

Table 9.1: Natural frequencies for beams with various end conditions.

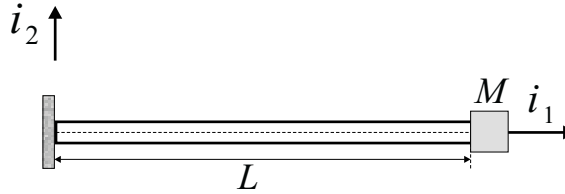


Figure 9.4: Cantilevered beam with a tip mass.

9.3.1 Problems

Problem 9.1

Derive the governing partial differential equations of motion for a shear deformable beam taking into account both translational and rotary inertial effects. Start from the expressions for the total strain and kinetic energies, then use Hamilton's Principle to derive the equations of motion. (Don't bother solving the resulting equations.)

Problem 9.2

Consider the cantilevered beam of length L with a tip mass M , as depicted in fig. 9.4. The beam has a uniform bending stiffness I_{33}^c and mass per unit span m .

1. Find the natural frequencies and mode shapes of the system. Use the following nondimensional mass parameter $\beta = M/(mL)$.
2. Check your results for $\beta = 0$ and $\beta \rightarrow \infty$; use the results tabulated in the notes.
3. Solve the same problem using an energy approach with the following assumed displacement mode $U_2(\eta, t) = (\eta^4 - 4\eta^3 + 6\eta^2) a(t)$. Compare your results with the exact frequencies obtained earlier. Quantify the error for $\beta = 0$ and 1.
4. On the same graph, plot the lowest natural frequency as a function of β for both exact and approximate solutions. Use a log scale, $\beta \in [10^{-03}, 10^{+03}]$.

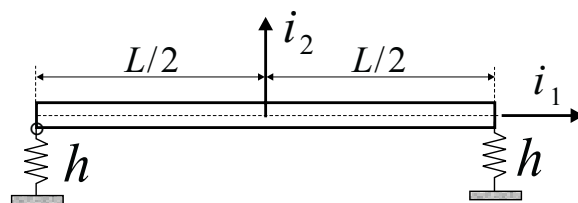


Figure 9.5: Rotor shaft supported on two end bearing.

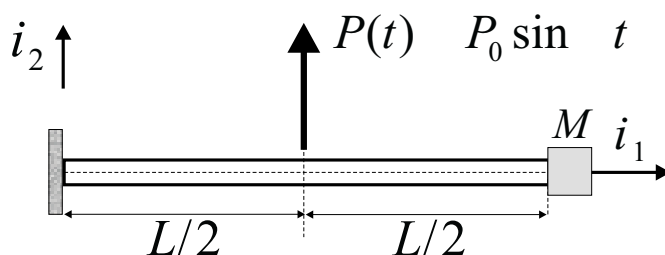


Figure 9.6: Cantilevered beam with a tip mass under a pulsating load.

Problem 9.3

Consider a rotor shaft of length L simply supported on two end bearings, as depicted in fig. 9.5. The stiffness of the bearings is represented by springs with stiffness constants h . Let $\eta = x_1/L$ and $h^* = hL^3/I_{33}^c$.

1. Find the natural frequencies of the system based on Euler-Bernoulli beam theory.
2. Check your results for $h^* = 0$ and $h^* \rightarrow \infty$; compare with the results tabulated in the notes.
3. Find an approximate solution of the problem using the following assumed modes: $U_2(x_1, t) = a_1(t) + a_2(t)\eta + a_3(t) \cos \pi\eta + a_4(t) \sin 2\pi\eta$.
4. Compare the exact and approximate solutions for $h^* = 10$. Discuss your results when $h^* = 0$ and $h^* \rightarrow \infty$. On the same graph, plot the lowest symmetric natural frequency as a function of h^* for both exact and approximate solutions. Plot for the anti-symmetric natural frequency. In both cases, use a log scale, $h^* \in [10^{-03}, 10^{+05}]$.

Hint: For both exact and approximate solutions, the problem splits into two separate parts: the symmetric and the anti-symmetric modes. This will ease the algebra considerably.

Problem 9.4

Consider the cantilevered beam of length L with a tip mass M as depicted in fig. 9.6. The tip mass is such that $\beta = M/(mL) = 1.0$. The beam is subjected to a mid-span pulsating load $P(t) = P_0 \sin \omega t$.

1. Find the natural frequencies and vibration modes of the system. Plot the lowest six eigenmodes of the system.

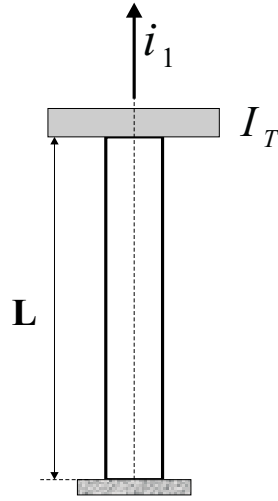


Figure 9.7: Thin-walled tube with a tip disk.

2. Find the response of the system using the modal superposition approach. Use the eigenmodes of the homogeneous system as the assumed modes.
3. Study the convergence of the procedure as the a number of modes used in the expansion increases. Plot the tip displacement $\bar{U}_2^{\text{tip}}$ versus $\tau \in [0, 15.0]$ for $\bar{\Omega} = 10.0$ using 2, 4 and 6 eigenmodes.
4. On the same graph, plot the time history of the transverse displacement of the beam at the mid- and tip-span locations, $\bar{U}_2^{\text{mid}}$ and $\bar{U}_2^{\text{tip}}$, respectively. Present graphs for $\bar{\Omega} = 1.2, 1.55, 15.0$ and 16.25 . Comment on your results.
5. Plot the time history of the bending moment at the root of the beam for the same excitation frequencies.

In all the plots, use the following nondimensional quantities

$$\tau = \sqrt{\frac{I_{33}^c}{mL^4}} t; \quad \bar{U}_2 = \frac{I_{33}^c U_2}{P_0 L^3}; \quad \bar{M}_3 = \frac{M_3}{P_0 L}.$$

Problem 9.5

The device shown in fig. 9.7 is used to measure the polar moment of inertia I_T of a disk. This disk is mounted on top of a thin-walled tube of length L , radius R_m , and thickness t . The tube is clamped at one end, and the disk is mounted at the other. The assembly is excited in torsion, and the lowest torsional frequency is measured. L , R_m , and t are given, as well as the tube material properties: shearing modulus G and material density ρ .

Find the relationship between the unknown polar moment of inertia and the measured natural frequency f in Hertz.

Problem 9.6

Find the frequency equation for the *axial vibrations* if the bar of length L with a tip mass M depicted in fig. 9.4. The bar has a uniform axial stiffness S and mass per unit span m . It will be

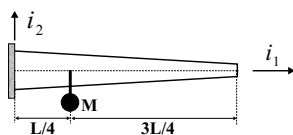


Figure 9.8: Cantilevered wing with quarter span nacelle.

convenient to define the nondimensional tip mass $\beta = M/(mL)$. Plot the lowest natural frequency of the system as a function of $\beta \in [0, 5]$.

Problem 9.7

The aircraft wing shown in fig. 9.8 above is clamped at the root and support an engine nacelle of mass M at quarter span. The wing is tapered in such a way that both bending stiffness, $I_{33}^c(\eta)$, and mass per unit span, $m(\eta)$, are linearly distributed along the wing span, *i.e.*

$$I_{33}^c = I_0 \left(1 - \frac{\eta}{2}\right); \quad m = m_0 \left(1 - \frac{\eta}{2}\right); \quad M = m_0 L,$$

where L is the span of the wing, and $\eta = x_1/L$. Find the lowest natural bending frequency of the wing assuming

$$U_2(\eta, t) = \eta^2 a_1(t) + (\eta^2 - \eta^3) a_2(t).$$

Problem 9.8

Consider a simply supported beam of span L subjected to a large axial load P that can be tensile or compressive. Use an energy approach to compute the lowest natural frequency of the system with the following assumed modes

$$U_2(x_1, t) = a_1(t) \sin \frac{\pi x_1}{L} + a_2(t) \sin \frac{2\pi x_1}{L}.$$

Plot the nondimensional natural frequencies $\omega_1/\omega_{\text{Euler}1}$ and $\omega_2/\omega_{\text{Euler}2}$ versus the nondimensional axial load $P/P_{\text{Euler}} \in [-1, 8]$, where P is taken positive in tension. What happens to the lowest natural frequency when $P/P_{\text{Euler}} = -1$? $\omega_{\text{Euler}1}$ and $\omega_{\text{Euler}2}$ are the two lowest natural frequencies of the beam when $P = 0$, and P_{Euler} the lowest buckling load of the system.

Chapter 10

Introduction to Composite Materials

10.1 Introduction

An important phase of structural design is the selection of a specific material. Table 10.1 lists the physical properties of three commonly used metals: aluminum, titanium, and steel. This table lists their respective ultimate stress, modulus of elasticity, and density. Table 10.2 lists the corresponding properties for a number of fibers.

Table 10.1 shows that the ultimate allowable stress and stiffness of steel are far superior to those of titanium or aluminum. Why then is steel not always preferred, since it is so far stronger and stiffer? A second look at table 10.1 shows that while steel is far stronger and stiffer, it is also far heavier than the other two metals. Clearly, in a weight sensitive design a compromise must be made between conflicting characteristics. In fact, it is important to compare the performance of these various materials for specific structural applications. Three categories of structural design situations will be investigated, namely *strength design*, *stiffness design*, and *buckling design*. A *performance index* of the material will be derived in each case.

1. Strength design.

Consider a sheet of material of length L , width b , and thickness t , subjected to a tension load P , as depicted in fig. 10.1. Assuming the stress distribution to be uniform over the cross-section of the sheet, the total load the material can carry is:

$$P_{max} = \sigma_{ult}bt. \quad (10.1)$$

	Ultimate stress [MPa]	Modulus of elasticity [GPa]	Density [kg/m ³]
Aluminum	620	73	2700
Titanium	1900	115	4700
Steel	4100	210	7700

Table 10.1: Physical properties of a few metals.

	Ultimate stress [MPa]	Modulus of elasticity [GPa]	Density [kg/m ³]
E-Glass	3400	72	2550
S-Glass	4800	86	2500
Carbon	1700	190	1410
Boron	3400	400	2570
Graphite	1700	250	1410

Table 10.2: Physical properties of a few fibers.

where σ_{ult} is the ultimate allowable stress for the material. The total mass M of the structure is:

$$M = \rho btL \quad (10.2)$$

where ρ is the material density. Eliminating the sheet thickness between eqs. (10.1) and (10.2) yields

$$P_{max} = \frac{M \sigma_{ult}}{L \rho}. \quad (10.3)$$

For a given mass and geometry of the structure, the maximum load it can carry is

$$P_{max} \propto \frac{\sigma_{ult}}{\rho}. \quad (10.4)$$

Clearly, σ_{ult}/ρ is the desired material performance index for a strength design.

2. Stiffness design.

In many instances the stiffness of a structure is specified, but more often than not, it is the natural frequency of the structure that must be maximized. Consider the cantilevered, thin-walled beam of length L consisting of two thin skins of width b and thickness t separated by a distance h as shown in fig. 10.1. The natural frequency of this structure is:

$$\omega \propto \frac{1}{L^2} \left[\frac{I_{22}^c}{m} \right]^{1/2} \quad (10.5)$$

where I_{22}^c is the bending stiffness, and m the mass per unit span of the beam. These quantities are readily found as:

$$I_{22}^c = 2Ebt \frac{h^2}{4} \left[1 + \frac{1}{3} \left(\frac{t}{h} \right)^2 \right]; \quad m = 2\rho bt. \quad (10.6)$$

For a thin-walled structure $t/h \ll 1$, and the natural frequency becomes:

$$\omega \propto \frac{h}{L^2} \left[\frac{E}{\rho} \right]^{1/2} \quad (10.7)$$

For a given configuration of the structure, h and L are given quantities, and clearly, $\sqrt{E/\rho}$ is the desired material performance index for stiffness design.

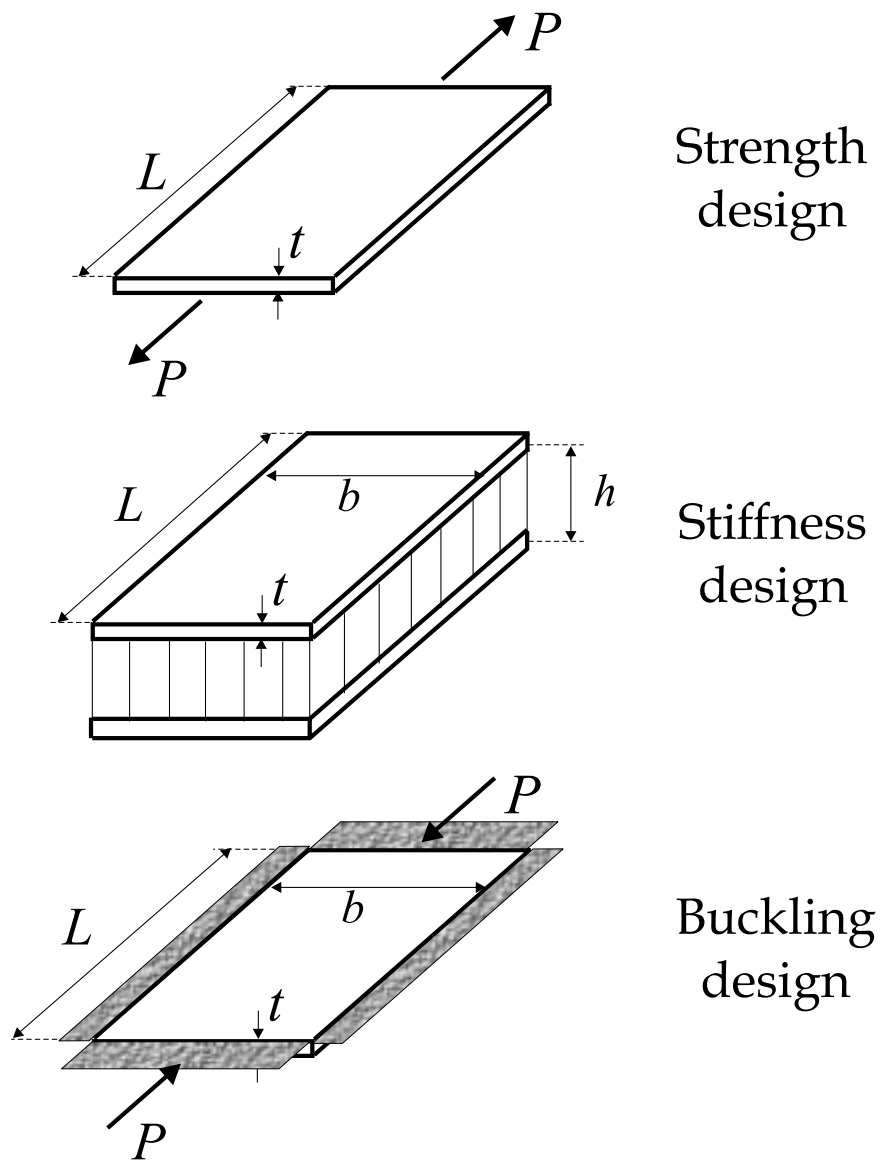


Figure 10.1: Three types of design situation.

Performance Index	Strength design σ_{ult}/ρ [$10^3 m^2/sec^2$]	Stiffness design $\sqrt{E/\rho}$ [$10^3 m/sec$]	Buckling design E/ρ^3 [$m^8/(kg^2 sec^2)$]
Aluminum	230	5.2	3.7
Titanium	405	4.9	1.1
Steel	530	5.2	0.46

Table 10.3: Structural design performance indices for a few metals.

3. Buckling design.

Fig. 10.1 shows a thin plate of length L , width b , and thickness t . The plate is simply supported around all its edges, and subjected to an in-plane compressive load P . The critical value of the load that will cause the plate to buckle is:

$$P_{cr} \propto \frac{Et^3}{b}. \quad (10.8)$$

The total mass of the structure is given by (10.2). Eliminating the thickness of the plate from (10.8) and (10.2) yields:

$$P_{cr} \propto \frac{M^3}{b^4 L^3} \frac{E}{\rho^3}. \quad (10.9)$$

For a given mass and geometry of the structure, the buckling load is proportional to E/ρ^3 , which is the desired performance index for buckling design.

Table 10.3 lists the performance indices σ_{ult}/ρ , $\sqrt{E/\rho}$, and E/ρ^3 for strength, stiffness, and buckling designs, respectively. Table 10.4 lists the corresponding quantities for a few fibers.

Consider first the data of table 10.3. Clearly steel is the most performant material for strength design. However, when it comes to stiffness design, the three metals perform about equally well, with only a slight disadvantage for titanium. Finally, comparing the strength and buckling designs, the ranking of the materials is now reversed: aluminum performs far better than steel and titanium in buckling design.

The same observations can be made about the fibers for which data is listed in table 10.4. In a strength design, S-glass outperforms the other fibers. The situation is reversed for stiffness and buckling designs. It is clear that the third power of the density in the denominator of the buckling design performance index makes lighter materials perform well in buckling sensitive designs.

It is not possible to directly compare the materials in tables 10.3 and 10.4. Indeed, the various metals can be used as structural materials, whereas the fibers cannot be used, as such, as structural materials. It is however clear that the remarkably high performance indices of these fibers justifies a closer look at their potential use in structural applications.

Performance Index	Strength design σ_{ult}/ρ [$10^3 m^2/sec^2$]	Stiffness design $\sqrt{E}/\rho$ [$10^3 m/sec$]	Buckling design E/ρ^3 [$m^8/(kg^2 sec^2)$]
E-Glass	1330	5.3	4.3
S-Glass	1920	5.9	5.5
Carbon	1200	11.6	68
Boron	1320	12.5	23
Graphite	1200	13.3	89

Table 10.4: Structural design performance indices for a few fibers.

10.2 Composite materials

Advanced composite materials for structural applications are made by embedding fibers that are all aligned in a single direction in a matrix material. A great variety of polymeric materials can be used as matrix materials. Thermoset materials such as epoxy have been extensively used as matrices for composite materials. The mechanical properties of epoxy are:

$$\sigma_{ult}^{tension} = 50 \text{ MPa}; \quad \sigma_{ult}^{compression} = 140 \text{ MPa}, \quad (10.10)$$

for the ultimate allowable stress in tension and compression, respectively. The modulus of elasticity, and the density of the material are:

$$E = 3.5 \text{ GPa}; \quad \rho = 1300 \text{ kg/m}^3. \quad (10.11)$$

A very crude way of approximating the strength of a composite material consisting of fibers all aligned in a single direction embedded in a matrix is to use a *rule of mixture*:

$$S_c = V_f S_f + V_m S_m, \quad (10.12)$$

where S_c , S_f , and S_m are the strength of the composite, fiber, and matrix materials, respectively; and V_f , and V_m the volume fractions of fiber, and matrix materials, respectively. If the material contains no voids $V_f + V_m = 1$.

Consider a composite material consisting of graphite fibers ($V_f = 0.6$), embedded in an epoxy matrix ($V_m = 0.4$). The strength of the composite can be estimated using eq. (10.12) and the data of table 10.2:

$$S_c = 1700 \times 0.6 + 50 \times 0.4 = 1020 + 20 = 1040 \text{ MPa}. \quad (10.13)$$

Clearly, the matrix material contributes very little to the strength of the composite.

The stiffness of the composite can also be crudely estimated from the following reasoning. Assume the various phases of the material to be bonded together perfectly, then

$$\varepsilon = \varepsilon_m = \varepsilon_f = \varepsilon_c, \quad (10.14)$$

where ε_m , ε_f , and ε_c are the strains in the matrix, fiber, and composite materials, respectively. If a sheet of this material is subjected to a tensile load P , the average stress in the composite σ_c can be defined as follows:

$$P = A_c \sigma_c = A_f \sigma_f + A_m \sigma_m, \quad (10.15)$$

where σ_f , and σ_m are the stresses in the fiber, and matrix materials, respectively; and A_c , A_f , and A_m the cross-sectional areas of composite, fiber, and matrix materials, respectively. Dividing eq. (10.15) by A_c yields

$$\sigma_c = \frac{A_f}{A_c}\sigma_f + \frac{A_m}{A_c}\sigma_m = V_f\sigma_f + V_m\sigma_m. \quad (10.16)$$

If both fiber and matrix materials are assumed to be linear elastic, isotropic materials, the following constitutive laws adequately describe their behavior:

$$\sigma_f = E_f\varepsilon_f; \quad \sigma_m = E_m\varepsilon_m, \quad (10.17)$$

where E_f , and E_m are the moduli of elasticity for the fiber, and matrix materials, respectively. Similarly, the modulus of elasticity E_c for the composite is defined as:

$$\sigma_c = E_c\varepsilon_c. \quad (10.18)$$

Introducing eqs. (10.17) and (10.18) into eq. (10.16), and taking into account the assumed equality of the strain in the various materials, eq. (10.14) yields

$$E_c = V_fE_f + V_mE_m. \quad (10.19)$$

For the graphite epoxy material considered above, the composite modulus of elasticity can be estimated as:

$$E_c = 250 \times 0.6 + 3.5 \times 0.4 = 150 + 1.4 \approx 150 \text{ GPa}. \quad (10.20)$$

Here again, the intrinsic stiffness of the matrix material contribute little to the stiffness of the composite.

The above discussion clearly shows that a significant fraction of the high strength and stiffness of the fibers is directly inherited by the composite. However, the matrix material contributes little to the strength and stiffness of the composite. This observation prompts the following question: what is the role of the matrix material in a composite? The matrix is needed to keep all the fibers together, and to provide an adequate surface finish. A less obvious role of the matrix is to diffuse the stresses among the otherwise isolated fibers. This aspect is explored in the next section.

10.3 Stress diffusion in a composite

Consider a lamina consisting of fibers all aligned in a single direction embedded in a matrix material. The lamina is subjected to a far field stress σ_0 . If all the fibers were continuous, it would be easy to see how the entire load would be carried by the fibers only, with no significant contribution of the matrix. However, in practical situations, numerous broken fibers will be present in the lamina. Fig. 10.2 shows the geometry of the problem, including a broken fiber of length $2L$. At the two ends of the fiber, the stress in the fiber must vanish. Nevertheless, the matrix material adjacent to the broken fiber will transfer stress from the

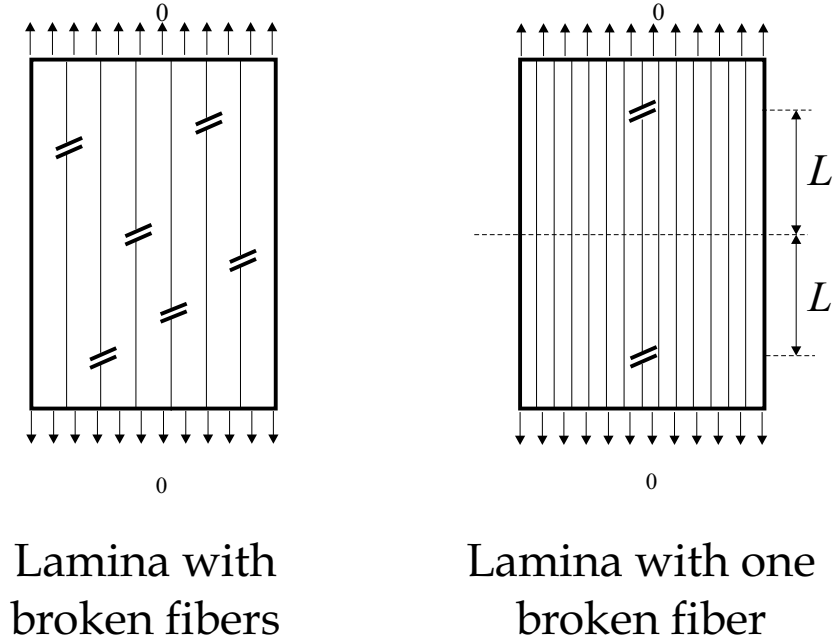


Figure 10.2: Lamina with a broken fiber.

surrounding material to the broken fiber. This stress diffusion process is a very important phenomenon as it allows *all fibers*, including broken fibers, to carry the applied load.

A simplified model of this phenomenon is depicted in fig. 10.3. It consists of a cylindrical fiber of radius r_f , surrounded by circular sleeve of matrix material of outer radius r_m , itself surrounded by a circular sleeve of composite material of outer radius r_a . The following assumptions will be made:

1. The matrix material carries shear stresses only. This assumption can be justified by the fact that the stiffness of the fiber is far greater than that of the matrix, and hence, the axial stress it carries is far greater than that carried by the matrix.
2. The axial stress in the fiber is uniformly distributed over its cross-section.
3. The properties of the composite material surrounding the core fiber / matrix cylinder can *smeared*, i.e. the existence of individual fibers can be ignored, except for the specific broken fiber at the heart of the model.
4. The various phases of the model are perfectly bonded together.

Due to the symmetry of the problem, the displacements at $x_1 = 0$ are set to zero. Fig. 10.4 shows the displacements $u_f(x_1)$, and $u_a(x_1)$ of the fiber, and composite, respectively. The stress-strain relationships for the various constituents of the model are:

$$\varepsilon_f = \frac{du_f}{dx_1}; \quad (10.21)$$

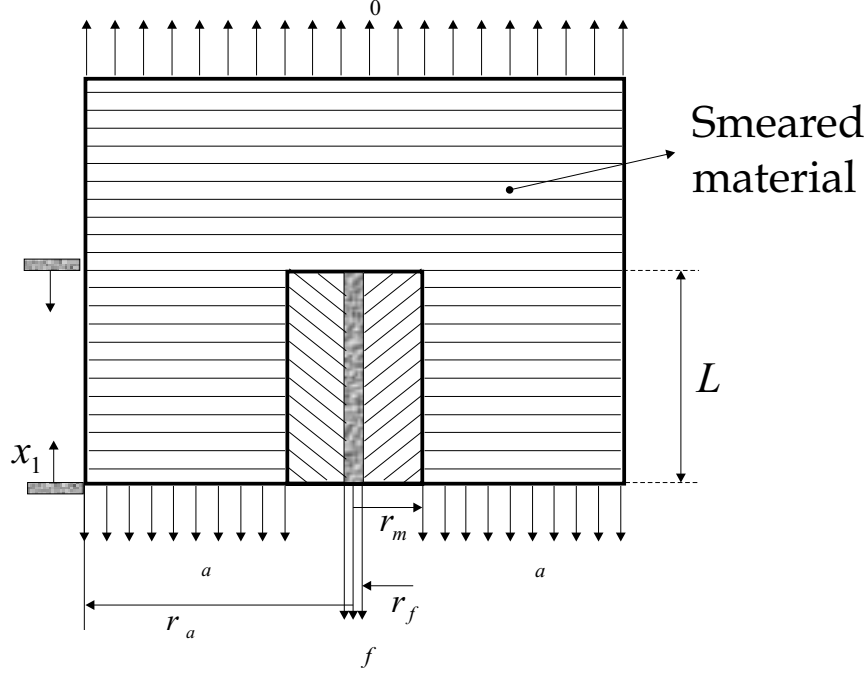


Figure 10.3: Simplified stress diffusion problem.

$$\varepsilon_a = \frac{du_a}{dx_1}; \quad (10.22)$$

$$\gamma_m = \frac{u_a - u_f}{r_m - r_f}, \quad (10.23)$$

where ε_f , ε_a , and γ_m are the axial strains in the fiber and composite, and the shearing strain in the matrix, respectively.

A free body diagram of a differential element of the fiber is shown in fig. 10.4. A summation of the forces along the axis of the fiber yields:

$$\frac{d\sigma_f}{dx_1} + \frac{2}{r_f}\tau_m = 0, \quad (10.24)$$

where $\sigma_f(x_1)$ is the uniform axial stress in the fiber, and $\tau_m(x_1)$ the shearing stress in the matrix. On the other hand, the free body diagram of the entire model depicted in fig. 10.3 yields an overall equilibrium equation

$$\sigma_a = \frac{1}{1 - (r_m/r_a)^2}\sigma_0 - \left(\frac{r_f}{r_a}\right)^2 \frac{1}{1 - (r_m/r_a)^2}\sigma_f \approx \sigma_0. \quad (10.25)$$

It is clear that the fiber has a much smaller radius than the overall composite, i.e. $r_f/r_a \ll 1$, and the second term of this equation become negligible. Furthermore, $r_m/r_a \ll 1$, i.e. $1 - (r_m/r_a)^2 \approx 1$.

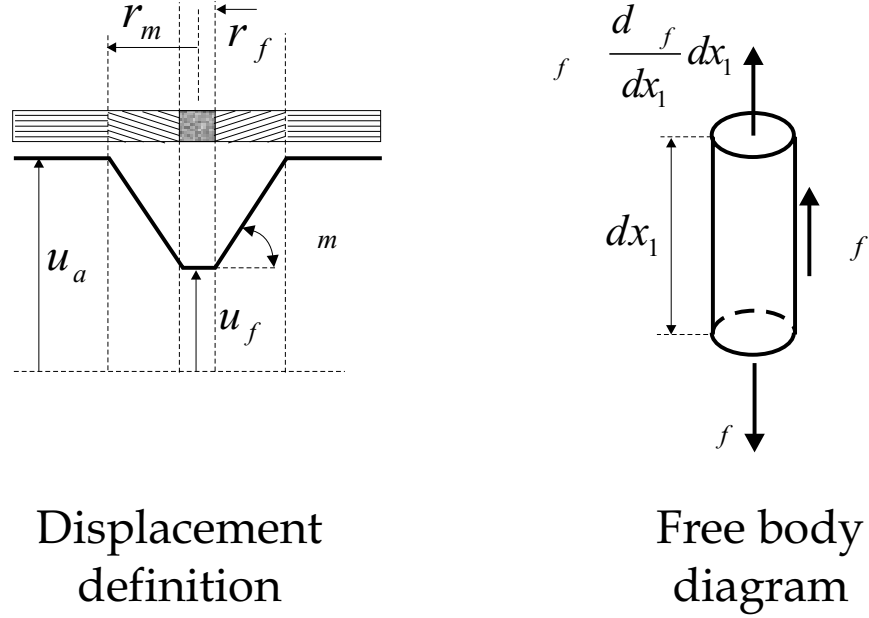


Figure 10.4: Displacement definition and free body diagram of a differential element of fiber.

Finally, the constitutive laws for the various constituents of the model are:

$$\sigma_f = E_f \varepsilon_f; \quad (10.26)$$

$$\sigma_a = E_a \varepsilon_a; \quad (10.27)$$

$$\tau_m = G_m \gamma_m; \quad (10.28)$$

where E_f , E_a , and G_m are the moduli of elasticity of the fiber, and composite, and the shearing modulus of the matrix, respectively.

Introducing the matrix material constitutive law, eq. (10.28), and the definition of the shearing strain, eq. (10.23) into the fiber equilibrium eq. (10.24) yields

$$\frac{d\sigma_f}{dx_1} + \frac{2G_m}{r_f(r_m - r_f)}(u_a - u_f) = 0.$$

Taking a derivative of this equation with respect to x_1 , introducing the definition of the fiber, and composite strains, eqs. (10.21), and (10.22), respectively, and using the constitutive laws, eqs. (10.26), and (10.27) leads to

$$\frac{d^2\sigma_f}{dx_1^2} + \frac{2G_m}{r_f(r_m - r_f)}\left(\frac{\sigma_a}{E_a} - \frac{\sigma_f}{E_f}\right) = 0.$$

Finally, the stress in the composite σ_a is eliminated by means of the overall equilibrium eq. (10.25) to find the governing equation for the fiber stress

$$\frac{d^2\sigma_f}{dx_1^2} - \frac{2}{r_f(r_m - r_f)} \frac{G_m}{E_f} \sigma_f = -\frac{2}{r_f(r_m - r_f)} \frac{G_m}{E_f} \frac{E_f}{E_a} \sigma_0.$$

The following non-dimensional variable η is introduced:

$$\eta = \frac{L - x_1}{2r_f}. \quad (10.29)$$

As shown in fig. 10.3, η measures the distance from the fiber break non dimensionalized by the fiber diameter. The governing equation for the fiber stress becomes

$$\sigma_f'' - \lambda^2 \sigma_f = -\lambda^2 \frac{E_f}{E_a} \sigma_0,$$

where the notation $(.)'$ is used to denote a derivative with respect to η ; and

$$\lambda^2 = 8 \frac{G_m}{E_f} \frac{r_f/r_m}{1 - r_f/r_m}.$$

The volume fraction of the material is $V_f = (\pi r_f^2)/(\pi r_m^2) = (r_f/r_m)^2$. Furthermore, the rule of mixture for the modulus of elasticity, eq. (10.19), yields

$$\frac{E_f}{E_a} = \frac{E_f}{V_f E_f + V_m E_m} \approx \frac{E_f}{V_f E_f} = \frac{1}{V_f},$$

where the fact that $E_m \ll E_f$ was taken into account. The governing equation finally write

$$\sigma_f'' - \lambda^2 \sigma_f = -\lambda^2 \frac{\sigma_0}{V_f}, \quad (10.30)$$

where

$$\lambda^2 = 8 \frac{G_m}{E_f} \frac{\sqrt{V_f}}{1 - \sqrt{V_f}}. \quad (10.31)$$

The boundary conditions are $\sigma_f = 0$ at the broken end of the fiber, i.e. @ $\eta = 0$, and @ $\eta = L/2r_f$, the symmetry of the problem requires $\sigma_f' = 0$. The solution of eq. (10.30) subjected to these boundary conditions is

$$\frac{\sigma_f}{\sigma_0} = \frac{1}{V_f} \left(1 - \frac{\cosh \lambda(L/2r_f - \eta)}{\cosh(\lambda L/2r_f)} \right) \approx \frac{1}{V_f} (1 - e^{-\lambda \eta}). \quad (10.32)$$

To illustrate the distribution of stress near a fiber break, three material systems will be considered. Table 10.5 lists the relevant parameters for the three material systems: Boron, Graphite, and Kevlar fibers in an epoxy matrix of shearing modulus of 1.35 *GPa*.

The fiber stress at a large distance from the fiber break can be obtained from eq. (10.16): $\sigma_0 = V_f \sigma_{f\infty} + (1 - V_f) \sigma_{m\infty} \approx V_f \sigma_{f\infty}$. The stress distribution, eq. (10.32), then writes

$$\frac{\sigma_f}{\sigma_{f\infty}} = 1 - e^{-\lambda \eta}, \quad (10.33)$$

where the notation $(.)_\infty$ is used to denote the value of the corresponding quantity at a large distance from the fiber break. The non dimensional parameter λ characterizes the fiber axial stress distribution near the fiber break, which is plotted in fig. 10.5 for the three material

Material system	Volume fraction	E_f [GPa]	λ Eq. (10.31)	δ/d_f Eq. (10.34)
Boron/Epoxy	0.5	400	0.255	11
Graphite/Epoxy	0.6	250	0.385	7.3
Kevlar/Epoxy	0.6	130	0.534	5.3

Table 10.5: Physical properties of three material systems.

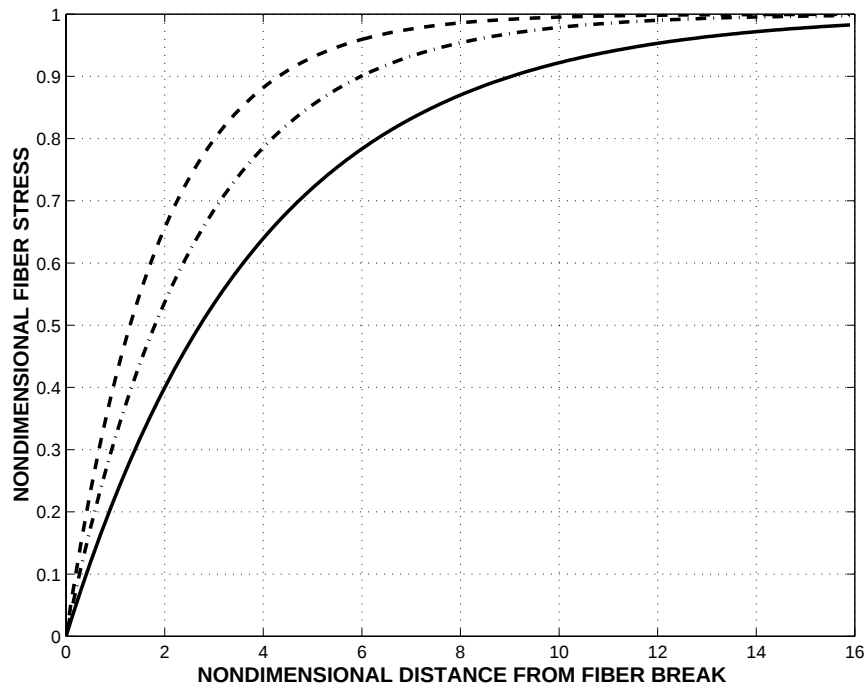


Figure 10.5: Distribution of fiber axial stress near a fiber break for three material systems.

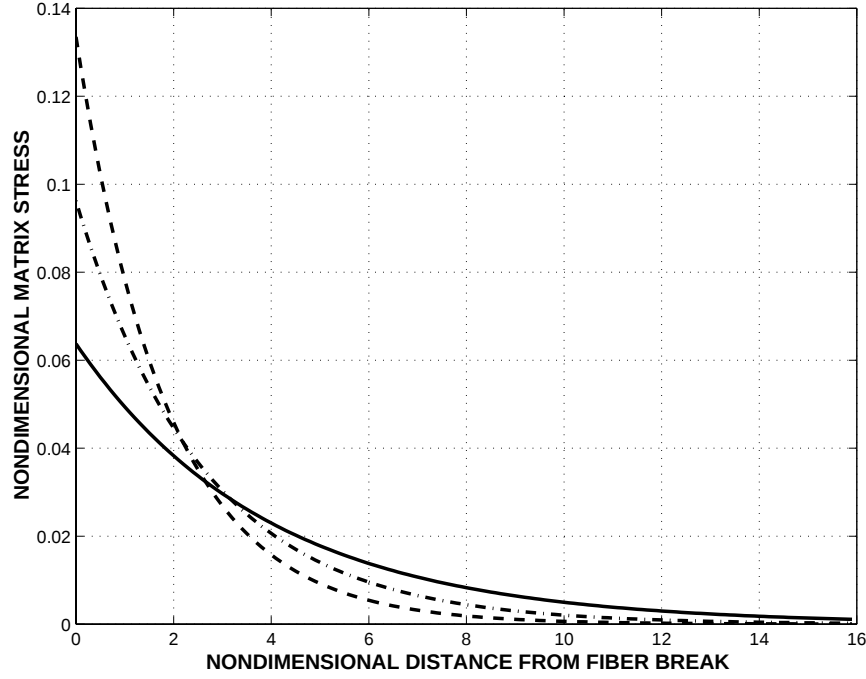


Figure 10.6: Distribution of matrix shear stress near a fiber break for three material systems.

systems. At $\eta = 0$, which corresponds to the fiber break, the fiber axial stress vanishes. However, the fiber axial stress rapidly grows to its far field value $\sigma_{f\infty}$.

It is convenient to define the fiber *ineffective length* δ as the distance it takes for the fiber stress to reach 95% of its far field value, i.e.

$$0.95 = 1 - e^{-\lambda\delta/d_f}.$$

where d_f is the fiber diameter. Solving this equation yields the ineffective length as

$$\frac{\delta}{d_f} \approx \left[\frac{E_f}{G_m} \frac{1 - \sqrt{V_f}}{\sqrt{V_f}} \right]^{1/2}. \quad (10.34)$$

The ineffective length can be thought of as the length of fiber, near a fiber break, that does not carry axial stress at full capacity. Table 10.5 lists the ineffective length for the three material systems. It appears that 5.3 fiber diameters away from a break, the Kevlar fiber is already carrying 95% of its far field stress. This means that the matrix material transfers the load from the surrounding material to the broken fiber *very rapidly*. This mechanism is called the *shear lag* mechanism because the shear stress in the matrix is effectively transferring the load to the fiber. The shear stress in the matrix can be readily evaluated from the fiber equilibrium equation (10.24) as

$$\frac{\tau_m}{\sigma_{f\infty}} = \frac{\lambda}{4} e^{-\lambda\eta}. \quad (10.35)$$

Fig. 10.6 shows the distribution of shear stress in the matrix near a fiber break, for the three material systems. The shear stress is maximum near the fiber break, then decays very rapidly.

An important role of the matrix material is now clearly apparent in the light of the above analysis. Near a fiber break, the matrix material transfer the stress from the surrounding material to the broken fiber. The shearing of the matrix near the fiber break is the mechanism that allows this stress transfer to occur. This mechanism is very efficient; for the material systems described above, the broken fiber is fully loaded by about ten fiber diameters from the fiber break. In other words, the zone affected by the fiber break is about 2δ in length (δ on each side of the break). For a graphite fiber with a diameter of 10 microns, the zone affected by a fiber break is only about 200 microns in length.

Another way of looking at this fact is to say that a fiber is continuous or *infinitely long*, if its total length is much larger, say 100 times larger than its ineffective length. Hence, a 10 micron diameter graphite fiber can be considered continuous or infinitely long when its length is $100 \times 100 \cdot 10^{-6} = 10 \cdot 10^{-3}m$. From a load carrying stand point, a 10 millimeter long graphite fiber can be considered continuous or infinitely long.

10.4 Constitutive laws for anisotropic materials

Constitutive laws characterize the mechanical behavior of materials by relating stresses and strains. The focus of this section is on the linear elastic behavior of anisotropic materials. The physical properties of anisotropic materials are directional, i.e. the physical response of the material depends on the direction in which it is acted upon. Consider, as an example, the stiffness of the unidirectional composite material described in section (10.2): in the fiber direction the stiffness of the composite is dominated by the high stiffness of the fiber (see eq. (10.19)). However, in the direction transverse to the fiber, the stiffness of the composite is dominated by that of the matrix material which is far smaller than that of the fiber. This contrasts with isotropic materials for which the mechanical response is identical in all directions.

The engineering strain components will be used to measure the straining of the material. These strain components are arranged in a vector form as

$$\underline{\varepsilon}^T = \left[\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_3}{\partial x_3}, \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}, \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}, \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right]. \quad (10.36)$$

The following engineering stress components will be used to measure the stress state in the material

$$\underline{\sigma}^T = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{23}, \tau_{13}, \tau_{12}]. \quad (10.37)$$

A linear elastic anisotropic material is characterized by the following relationships:

$$\underline{\sigma} = C\underline{\varepsilon}; \quad \underline{\varepsilon} = S\underline{\sigma}, \quad (10.38)$$

where C is a 6 by 6 stiffness matrix and S a 6 by 6 compliance matrix. Clearly, these two matrices are the inverse of each other, i.e.

$$C^{-1} = S. \quad (10.39)$$

The strain energy W stored in a differential element of the material is:

$$W = \frac{1}{2} \underline{\varepsilon}^T \underline{\sigma} = \frac{1}{2} \underline{\varepsilon}^T C \underline{\varepsilon} = \frac{1}{2} \underline{\sigma}^T S \underline{\sigma}. \quad (10.40)$$

The stored strain energy is a positive quantity for whatever deformation or stress state the material is subjected to. This implies that both stiffness and compliance matrices are symmetric and definite positive.

In general, the 6 by 6 stiffness matrix has $6 \times 6 = 36$ independent coefficients. However, the symmetry requirement reduces the number of independent coefficients to 21. The stress strain relationship eq. (10.38) written in expanded form is

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}, \quad (10.41)$$

where the entries in the lower triangular part of the stiffness matrix are equal to the corresponding upper triangular entries. The 21 constants C_{ij} characterize the behavior of the material. Each constant must be determined experimentally. A material characterized by relationship eq. (10.41) is called an *anisotropic* or *triclinic* material.

Materials sometimes possess a plane of symmetry. Let the plane $\mathbf{i}_1 \mathbf{i}_2$ be a plane of symmetry of the material. The stress-strain relationship reduces to:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}. \quad (10.42)$$

The stiffness coefficient C_{14} must be zero. Indeed, if C_{14} were not zero, an axial strain ε_{11} would give rise to a stress τ_{23} . However, the presence of a shearing stress τ_{23} would violate the symmetry of the response which is a natural consequence of the material symmetry. A systematic application of this symmetry argument shows that 8 coefficients must vanish, leaving $21 - 8 = 13$ independent coefficients. This type of material is called a *monoclinic* material.

Some materials show a higher level of symmetry characterized by two mutually orthogonal planes of symmetry, for instance the $\mathbf{i}_1 \mathbf{i}_2$ and $\mathbf{i}_2 \mathbf{i}_3$ planes. The stress strain relationship now writes:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}. \quad (10.43)$$

Here again symmetry arguments were used to prove that 12 coefficients of the stiffness matrix must vanish, leaving $21 - 12 = 9$ independent coefficients. This type of material is called an *orthotropic* material.

A case of particular importance to the study of laminated composite materials is that of materials presenting two orthogonal planes of symmetry, and one plane of isotropy. Let $\mathbf{i}_1 \mathbf{i}_2$ and $\mathbf{i}_2 \mathbf{i}_3$ be the mutually orthogonal planes of symmetry, and let the material be isotropic in the $\mathbf{i}_2 \mathbf{i}_3$ plane. This means, for instance, that the coefficients C_{12} and C_{13} should be identical due to the isotropic response of the material in the $\mathbf{i}_2 \mathbf{i}_3$ plane. The stress strain relationship now reduces to:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{22} & 0 & 0 & 0 \\ & & & \frac{C_{22} - C_{23}}{2} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}. \quad (10.44)$$

Five constants only are left for this material called *transversely isotropic*.

Finally, an *isotropic* material is characterized by an identical response in all directions, leading to the following stress strain relationship:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{C_{11} - C_{12}}{2} & 0 & 0 \\ & & & & \frac{C_{11} - C_{12}}{2} & 0 \\ & & & & & \frac{C_{11} - C_{12}}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}. \quad (10.45)$$

Two independent constants only are left for this type of material.

Relations (10.41) to (10.45) give the structure of the stiffness matrix for various types of materials. The compliance matrix can be obtained by inversion, and shows the same structure.

The actual structure of the stiffness matrix was obtained based on energy and symmetry arguments. An isotropic material was shown to be characterized by two independent coefficients C_{11} and C_{12} . However, an isotropic material is generally characterized by its Young's modulus and Poisson's ratio. These constants are called the *engineering constants*, because they can be readily measured experimentally. For the various types of materials, the stiffness and compliance terms can be expressed in terms of the engineering constants. The following section discusses the experimental determination and the physical interpretation of the engineering constants for a lamina made of unidirectional fibers embedded in a matrix material.

10.5 Constitutive laws for a lamina in the $\mathcal{S}^*$ system

Consider a thin sheet of composite material made of unidirectional fibers embedded in a matrix. Let the axis $\mathbf{i}_1^*$ be oriented along the fiber direction, $\mathbf{i}_2^*$ in the transverse direction, and $\mathbf{i}_3^*$ is perpendicular to the plane of the thin sheet. These axes form a basis denoted $\mathcal{S}^*$. If the diameter of the fiber is small compared to the thickness of the sheet, the material can be assumed to be a homogeneous, transversely isotropic material. In other words, the existence of individual fiber can be ignored; fibers and matrix materials are *smeared* into an equivalent, homogeneous material. For a linear elastic, transversely isotropic material the constitutive laws reduce to (10.44).

For numerous practical applications, the thin sheet of material can be assumed to be in a plane state of stress, i.e. $\sigma_{33}^* \approx \tau_{13}^* \approx \tau_{23}^* \approx 0$. The constitutive laws expressed in compliance form write:

$$\begin{bmatrix} \varepsilon_{11}^* \\ \varepsilon_{22}^* \\ \gamma_{12}^* \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1^*} & -\frac{\nu_{21}^*}{E_2^*} & 0 \\ -\frac{\nu_{12}^*}{E_1^*} & \frac{1}{E_2^*} & 0 \\ 0 & 0 & \frac{1}{G_{12}^*} \end{bmatrix} \begin{bmatrix} \sigma_{11}^* \\ \sigma_{22}^* \\ \tau_{12}^* \end{bmatrix}. \quad (10.46)$$

The notation $(.)^*$ is used to express quantities measured in the $\mathcal{S}^*$ system which is aligned with the fiber direction. The compliance matrix was expressed in terms of four constants E_1^* , E_2^* , ν_{12}^* , and G_{12}^* which are called the *engineering constants*. Note that the compliance matrix must be symmetric, thus $\nu_{12}^*/E_1^* = \nu_{21}^*/E_2^*$. This means that though five constants appear in the expression of the compliance matrix, one of them say ν_{21}^* , can be computed from the other, and hence, is not an independent quantity.

The engineering constants can be readily measured experimentally. Consider a simple test where the composite is subjected to a known stress σ_{11}^* in the fiber direction only, i.e. $\sigma_{22}^* = \tau_{12}^* = 0$, as depicted in fig. 10.7. The first equation of (10.46) now reduces to $\varepsilon_{11}^* = \sigma_{11}^*/E_1^*$. The strain ε_{11}^* in the fiber direction can be measured by means of a strain gauge, then the modulus of elasticity is computed as $E_1^* = \sigma_{11}^*/\varepsilon_{11}^*$. Clearly E_1^* is the modulus of elasticity of the material in the fiber direction. The second equation of (10.46) reads $\varepsilon_{22}^* = -\nu_{12}^*\sigma_{11}^*/E_1^*$. The strain ε_{22}^* in the direction transverse to the fiber can also be measured by means of a strain gauge. The Poisson's ratio can now be computed as $\nu_{12}^* = -E_1^*\varepsilon_{22}^*/\sigma_{11}^*$.

Consider now a second test where the composite material is subjected to a known stress σ_{22}^* in the direction transverse to the fiber only, i.e. $\sigma_{11}^* = \tau_{12}^* = 0$, as depicted in fig. 10.7. A measurement of the strain ε_{22}^* will then yield E_2^* , the modulus of elasticity of the material in the direction transverse to the fiber. An additional measurement of ε_{11}^* would yield ν_{21}^* . The symmetry of the compliance matrix can be verified experimentally by checking that the various measured quantities satisfy the symmetry condition $\nu_{12}^*/E_1^* = \nu_{21}^*/E_2^*$, within the expected experimental errors.

Finally, in a last test, the composite material is subjected to a known shear stress τ_{12}^* only, i.e. $\sigma_{11}^* = \sigma_{22}^* = 0$, as depicted in fig. 10.7. The last equation of (10.46) reduces to $\gamma_{12}^* = \tau_{12}^*/G_{12}^*$. A measurement of the shearing strain γ_{12}^* would then allow the evaluation of the shearing modulus $G_{12}^* = \tau_{12}^*/\gamma_{12}^*$.

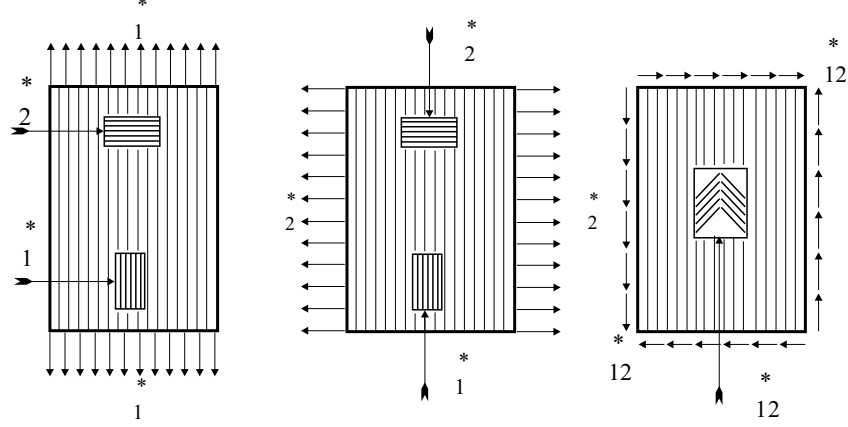


Figure 10.7: Three simple tests for the determination of the engineering constants.

The stiffness matrix is now obtained by inverting eq. (10.46) to find:

$$\begin{bmatrix} \sigma_{11}^* \\ \sigma_{22}^* \\ \tau_{12}^* \end{bmatrix} = \begin{bmatrix} \frac{E_1^*}{1 - \nu_{12}^{*2} E_2^*/E_1^*} & \frac{\nu_{12}^* E_2^*}{1 - \nu_{12}^{*2} E_2^*/E_1^*} & 0 \\ \frac{\nu_{12}^* E_2^*}{1 - \nu_{12}^{*2} E_2^*/E_1^*} & \frac{E_2^*}{1 - \nu_{12}^{*2} E_2^*/E_1^*} & 0 \\ 0 & 0 & G_{12}^* \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^* \\ \varepsilon_{22}^* \\ \gamma_{12}^* \end{bmatrix}. \quad (10.47)$$

To simplify the writing of this relationship, the following notations are introduced

$$\underline{\sigma}^* = \begin{bmatrix} \sigma_{11}^* \\ \sigma_{22}^* \\ \tau_{12}^* \end{bmatrix}; \quad \underline{\varepsilon}^* = \begin{bmatrix} \varepsilon_{11}^* \\ \varepsilon_{22}^* \\ \gamma_{12}^* \end{bmatrix}. \quad (10.48)$$

The material stiffness matrix in the fiber axis system $\mathcal{S}^*$ is defined as

$$C^* = \begin{bmatrix} \frac{E_1^*}{1 - \nu_{12}^{*2} E_2^*/E_1^*} & \frac{\nu_{12}^* E_2^*}{1 - \nu_{12}^{*2} E_2^*/E_1^*} & 0 \\ \frac{\nu_{12}^* E_2^*}{1 - \nu_{12}^{*2} E_2^*/E_1^*} & \frac{E_2^*}{1 - \nu_{12}^{*2} E_2^*/E_1^*} & 0 \\ 0 & 0 & G_{12}^* \end{bmatrix} = \begin{bmatrix} C_{11}^* & C_{12}^* & 0 \\ C_{12}^* & C_{22}^* & 0 \\ 0 & 0 & C_{66}^* \end{bmatrix}. \quad (10.49)$$

Finally, the constitutive law, eq. (10.47), can be written in the following compact form

$$\underline{\sigma}^* = C^* \underline{\varepsilon}^*. \quad (10.50)$$

Similarly, the material compliance matrix in the fiber axis system $\mathcal{S}^*$ is defined as

$$S^* = \begin{bmatrix} \frac{1}{E_1^*} & -\frac{\nu_{21}^*}{E_2^*} & 0 \\ -\frac{\nu_{12}^*}{E_1^*} & \frac{1}{E_2^*} & 0 \\ 0 & 0 & \frac{1}{G_{12}^*} \end{bmatrix} = \begin{bmatrix} S_{11}^* & S_{12}^* & 0 \\ S_{12}^* & S_{22}^* & 0 \\ 0 & 0 & S_{66}^* \end{bmatrix}. \quad (10.51)$$

Material type	V_f	E_1^* [GPa]	E_2^* [GPa]	ν_{12}^*	G_{12}^* [GPa]	density [kg/m ³]
Graphite/Epoxy T300/5208	0.70	180.	10.	0.28	7.0	1600
Graphite/Epoxy AS/3501	0.66	138.	9.	0.30	7.0	1600
Boron/Epoxy T300/5208	0.50	204.	18.	0.23	5.6	2000
Scotchply 1002	0.45	39.	8.	0.26	4.0	1800
Kevlar 49	0.60	76.	5.5	0.34	2.3	1460

Table 10.6: Engineering constants for lamina made of different materials.

The constitutive law (10.46) in compact form becomes:

$$\underline{\varepsilon}^* = S^* \underline{\sigma}^*. \quad (10.52)$$

The engineering constant for lamina made of a few different type of materials are listed in table 10.6. This table lists the volume fraction V_f , engineering constants E_1^* , E_2^* , ν_{12}^* , and G_{12}^* , as well as the density of the various lamina.

10.6 Constitutive laws for a lamina in the $\mathcal{S}$ system

In the previous section, the constitutive laws for a lamina of transversely isotropic lamina were discussed. The stresses and strains were measured in the fiber axis system $\mathcal{S}^*$. In many cases, however, the constitutive laws for the material are required in a direction that might not coincide with that of the fibers. Fig. 10.8 shows a reference axis system $\mathcal{S}$ defined by unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, and a transversely isotropic lamina with its fiber axis system $\mathcal{S}^*$. The fibers run at an angle θ with respect to a reference system. The angle θ is counted positive in the counter clockwise direction. Let $\underline{\sigma}$, and $\underline{\varepsilon}$ be the in-plane stresses and strains, respectively, in the reference system $\mathcal{S}$. The lamina constitutive laws in the $\mathcal{S}$ system now write:

$$\underline{\sigma} = C \underline{\varepsilon}. \quad (10.53)$$

Clearly, the stiffness matrix C could be obtained experimentally by performing a series a tests on the lamina, applying a stress in the $\mathbf{i}_1$ direction first, then in the $\mathbf{i}_2$ direction, as described in the previous section. Though conceptually feasible, this approach is not practical since a series of tests has to be performed each time the constitutive laws are desired at a specific angle θ . A better approach would be to relate the stiffness properties at an angle θ to those measured in the fiber direction. This can be readily achieved with the help of the formulae for the rotation of stresses and strains.

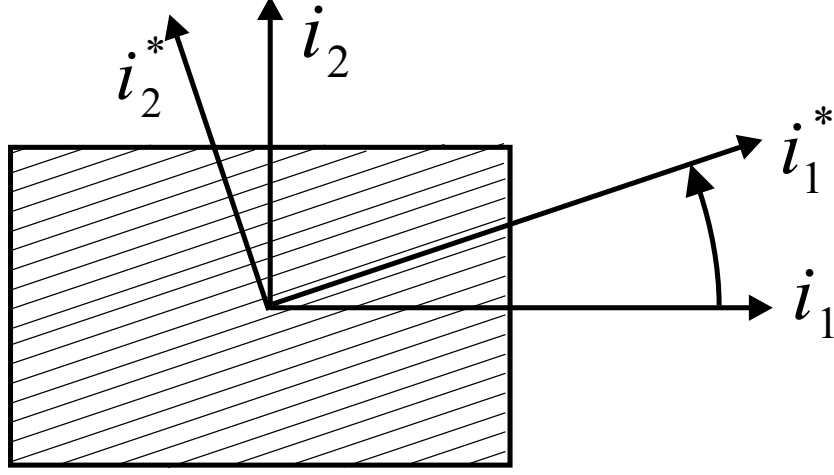


Figure 10.8: Definition of two axis systems for a lamina.

10.6.1 Rotation of constitutive laws

All the elements required to relate the constitutive laws in two different coordinate systems are now in place. The constitutive law for a layer expressed in the fiber system (10.50) is the starting point of the development:

$$\underline{\sigma}^* = C^* \underline{\varepsilon}^*,$$

Introducing the rotation formulae for stresses (1.27) and strain (1.68) yields:

$$\begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = C^* \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

The rotation matrix for stresses can be inverted to find:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} C^* \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}.$$

These constitutive laws now relate the stresses and strains in the $\mathcal{S}$ system and were also written as (10.53). Identifying these two relationships yields the components of the stiffness matrix in the reference system as

$$C = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} C^* \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix}. \quad (10.54)$$

Performing this triple matrix multiplication yields the various terms of the stiffness matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}, \quad (10.55)$$

where:

$$\begin{aligned}
C_{11} &= m^4 C_{11}^* + n^4 C_{22}^* + 2m^2 n^2 C_{12}^* + 4m^2 n^2 C_{66}^*; \\
C_{22} &= n^4 C_{11}^* + m^4 C_{22}^* + 2m^2 n^2 C_{12}^* + 4m^2 n^2 C_{66}^*; \\
C_{12} &= m^2 n^2 C_{11}^* + m^2 n^2 C_{22}^* + (m^4 + n^4) C_{12}^* - 4m^2 n^2 C_{66}^*; \\
C_{66} &= m^2 n^2 C_{11}^* + m^2 n^2 C_{22}^* - 2m^2 n^2 C_{12}^* + (m^2 - n^2)^2 C_{66}^*; \\
C_{16} &= m^3 n C_{11}^* - m n^3 C_{22}^* + (m n^3 - m^3 n) C_{12}^* + 2(m n^3 - m^3 n) C_{66}^*; \\
C_{26} &= m n^3 C_{11}^* - m^3 n C_{22}^* + (m^3 n - m n^3) C_{12}^* + 2(m^3 n - m n^3) C_{66}^*;
\end{aligned} \tag{10.56}$$

Here again, an alternate way of writing this transformation can be found by using the following trigonometric identities:

$$\cos^4 \theta = \frac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3); \quad \sin^4 \theta = \frac{1}{8}(\cos 4\theta - 4 \cos 2\theta + 3); \tag{10.57}$$

$$\cos^2 \theta \sin^2 \theta = \frac{1}{8}(1 - \cos 4\theta); \tag{10.58}$$

$$\cos^3 \theta \sin \theta = \frac{1}{8}(\sin 4\theta + 2 \sin 2\theta); \quad \cos \theta \sin^3 \theta = \frac{1}{8}(-\sin 4\theta + 2 \sin 2\theta). \tag{10.59}$$

Introducing these relationships into (10.56), then regrouping the trigonometric functions yields:

$$\begin{aligned}
C_{11} &= \alpha_1 + \alpha_2 + \alpha_3 \cos 2\theta + \alpha_4 \cos 4\theta; \\
C_{22} &= \alpha_1 + \alpha_2 - \alpha_3 \cos 2\theta + \alpha_4 \cos 4\theta; \\
C_{12} &= \alpha_1 - \alpha_2 - \alpha_4 \cos 4\theta; \\
C_{66} &= \alpha_2 - \alpha_4 \cos 4\theta; \\
C_{16} &= \frac{1}{2} \alpha_3 \sin 2\theta + \alpha_4 \sin 4\theta; \\
C_{26} &= \frac{1}{2} \alpha_3 \sin 2\theta - \alpha_4 \sin 4\theta;
\end{aligned} \tag{10.60}$$

where the material invariants α_1 , α_2 , α_3 , and α_4 , are defined as

$$\alpha_0 = 1 - \nu_{12}^* \frac{E_2^*}{E_1^*}; \tag{10.61}$$

$$\alpha_1 = \frac{1}{4\alpha_0}(E_1^* + E_2^* + 2\nu_{12}^* E_2^*); \quad \alpha_2 = \frac{1}{8\alpha_0}(E_1^* + E_2^* - 2\nu_{12}^* E_2^*) + \frac{G_{12}^*}{2}; \tag{10.62}$$

$$\alpha_3 = \frac{1}{2\alpha_0}(E_1^* - E_2^*); \quad \alpha_4 = \frac{1}{8\alpha_0}(E_1^* + E_2^* - 2\nu_{12}^* E_2^*) - \frac{G_{12}^*}{2}. \tag{10.63}$$

In summary, the stiffness matrix for a lamina can be obtained as follows. First, the engineering constants E_1^* , E_2^* , ν_{12}^* , and G_{12}^* are obtained experimentally by performing a series of test on the lamina (see section (10.5)). The stiffness matrix C^* in the fiber axis system $\mathcal{S}^*$ is then readily computed (see eq. (10.49)). The material invariants α_1 , α_2 , α_3 , and α_4 are computed from the engineering constants with the help of eqs.(10.62) and (10.63). Finally, the components of the stiffness matrix C in a reference axis system $\mathcal{S}$ can be evaluated for any angle θ for eqs. (10.60).

The material invariants for lamina made of a few different type of materials are listed in table 10.7. This table lists the material invariants computed from eqs. (10.62) and (10.63) based on the data of table 10.6.

Material type	α_1 [GPa]	α_2 [GPa]	α_3 [GPa]	α_4 [GPa]
Graphite/Epoxy T300/5208	49.11	26.65	85.37	19.65
Graphite/Epoxy AS/3501	38.32	21.30	64.88	14.30
Boron/Epoxy T300/5208	57.84	29.64	93.44	24.04
Scotchply 1002	12.97	7.43	15.72	3.43
Kevlar 49	21.49	10.95	35.55	8.65

Table 10.7: Material invariants for lamina made of different materials.

Fig. 10.9 shows the stiffness components C_{11} and C_{22} as a function of the lamina angle θ for the Graphite/Epoxy T300/5208 material. Note the rapid decline of the stiffness coefficient C_{11} when the lamina angle moves away from 0 degrees. This sharp decline is due to the high directionality of the lamina stiffness properties. The shearing stiffness components C_{66} is shown in fig. 10.10. The shearing stiffness drastically increases when the lamina angle is 45 degrees. This can be explained as follows: a state of pure shear is equivalent to stresses in tension and compression acting at 45 and 135 degree angles, respectively. These stresses are now aligned with the fiber direction which presents very high stiffness.

It is important to note that the terms C_{16} and C_{26} are non zero. These terms express a coupling between extension and shearing of the lamina. In contrast, the stiffness matrix C^* in the fiber system $\mathcal{S}^*$ has zero terms at the corresponding entries. Indeed, an extension shearing coupling term would violate the basic symmetry of the lamina in the $\mathcal{S}^*$ system. No such symmetry exists in the $\mathcal{S}$ system, and indeed a coupled response of the lamina is intuitively expected. Fig. 10.11 shows the stiffness components C_{16} and C_{26} as a function of lamina angle θ .

The lamina constitutive laws can be expressed in the stiffness form (10.53) or in the compliance form as

$$\underline{\varepsilon} = S \underline{\sigma}, \quad (10.64)$$

where S is the compliance matrix in the $\mathcal{S}$ system. Of course, the compliance matrix can be obtained by inverting the stiffness matrix, as indicated by (10.39). However, it can also be obtained directly. Introducing the stress rotation formula eq. (1.27) and strain rotation formula eq. (1.68) into eq. (10.52), and identifying the result with eq. (10.64) yields

$$S = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2 - n^2 \end{bmatrix} S^* \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix}. \quad (10.65)$$

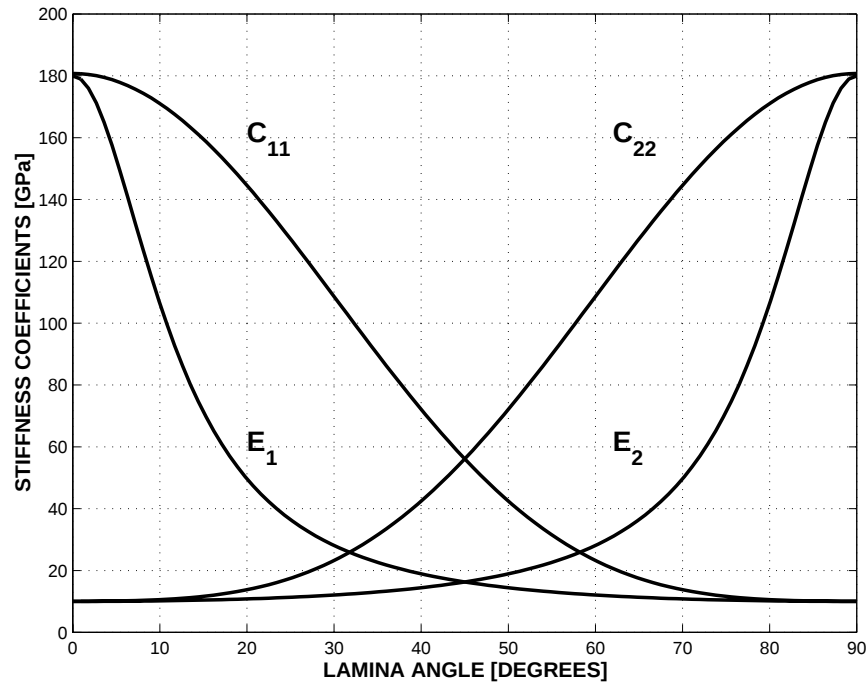


Figure 10.9: Variation of the stiffness coefficients C_{11} and C_{22} , and the engineering constants E_1 and E_2 with the lamina angle θ .

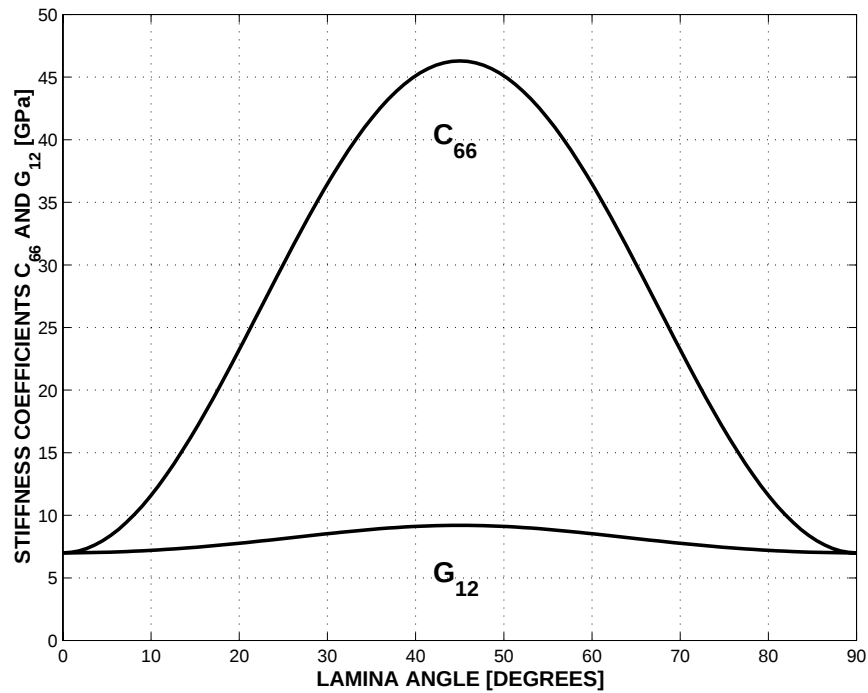


Figure 10.10: Variation of the stiffness coefficient C_{66} , and the engineering constants G_{12} with the lamina angle θ .

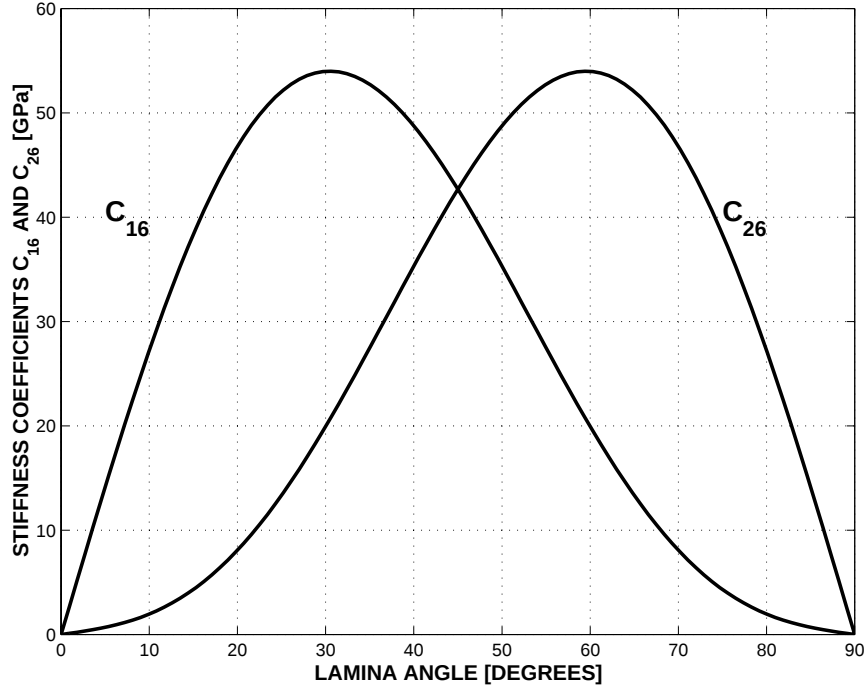


Figure 10.11: Variation of the stiffness coefficients C_{16} and C_{26} with the lamina angle θ .

Performing this triple matrix multiplication yields the terms of the compliance matrix:

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & \frac{\nu_{61}}{G_{12}} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{\nu_{62}}{G_{12}} \\ \frac{\nu_{16}}{E_1} & \frac{\nu_{26}}{E_2} & \frac{1}{G_{12}} \end{bmatrix}, \quad (10.66)$$

where

$$\begin{aligned} S_{11} &= m^4 S_{11}^* + n^4 S_{22}^* + 2m^2 n^2 S_{12}^* + m^2 n^2 S_{66}^*; \\ S_{22} &= n^4 S_{11}^* + m^4 S_{22}^* + 2m^2 n^2 S_{12}^* + m^2 n^2 S_{66}^*; \\ S_{12} &= m^2 n^2 S_{11}^* + m^2 n^2 S_{22}^* + (m^4 + n^4) S_{12}^* - m^2 n^2 S_{66}^*; \\ S_{66} &= 4m^2 n^2 S_{11}^* + 4m^2 n^2 S_{22}^* - 8m^2 n^2 S_{12}^* + (m^2 - n^2)^2 S_{66}^*; \\ S_{16} &= 2m^3 n S_{11}^* - 2mn^3 S_{22}^* + 2(mn^3 - m^3 n) S_{12}^* + (mn^3 - m^3 n) S_{66}^*; \\ S_{26} &= 2mn^3 S_{11}^* - 2m^3 n S_{22}^* + 2(m^3 n - mn^3) S_{12}^* + (m^3 n - mn^3) S_{66}^*; \end{aligned} \quad (10.67)$$

With the help of the trigonometric identities (10.57) to (10.59) these transformations laws can be written as:

$$\begin{aligned} S_{11} &= \beta_1 + \beta_2 + \beta_3 \cos 2\theta + \beta_4 \cos 4\theta; \\ S_{22} &= \beta_1 + \beta_2 - \beta_3 \cos 2\theta + \beta_4 \cos 4\theta; \\ S_{12} &= \beta_1 - \beta_2 - \beta_4 \cos 4\theta; \\ S_{66} &= 4\beta_2 - 4\beta_4 \cos 4\theta; \\ S_{16} &= \beta_3 \sin 2\theta + 2\beta_4 \sin 4\theta; \\ S_{26} &= \beta_3 \sin 2\theta - 2\beta_4 \sin 4\theta; \end{aligned} \quad (10.68)$$

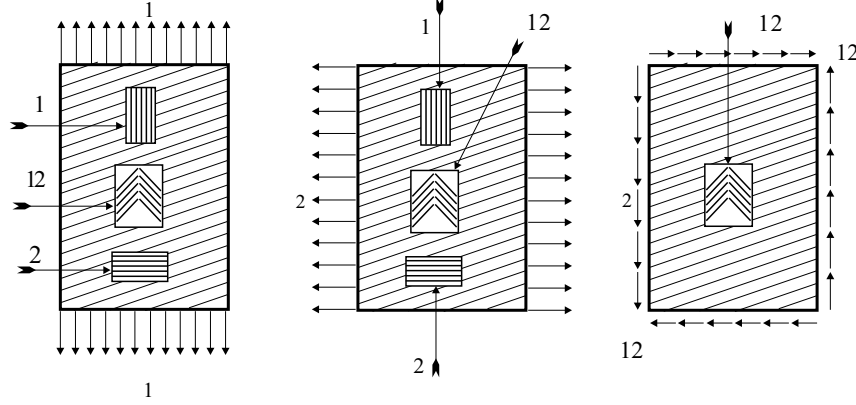


Figure 10.12: Three simple tests for the determination of the engineering constants.

where the material invariants β_1 , β_2 , β_3 , and β_4 , are defined as

$$\beta_1 = \frac{1}{4} \left(\frac{1}{E_1^*} + \frac{1}{E_2^*} - \frac{2\nu_{12}^*}{E_1^*} \right); \quad \beta_2 = \frac{1}{8} \left(\frac{1}{E_1^*} + \frac{1}{E_2^*} + \frac{2\nu_{12}^*}{E_1^*} \right) + \frac{1}{8G_{12}^*}; \quad (10.69)$$

$$\beta_3 = \frac{1}{2} \left(\frac{1}{E_1^*} - \frac{1}{E_2^*} \right); \quad \beta_4 = \frac{1}{8} \left(\frac{1}{E_1^*} + \frac{1}{E_2^*} + \frac{2\nu_{12}^*}{E_1^*} \right) - \frac{1}{8G_{12}^*}. \quad (10.70)$$

Equation (10.66) defines E_1 , E_2 , ν_{12} , G_{12} , ν_{16} , and ν_{26} , the engineering constants in the $\mathcal{S}$ system. Due to the symmetry of the compliance matrix, the following relationships hold $\nu_{12}/E_1 = \nu_{21}/E_2$, $\nu_{16}/E_1 = \nu_{61}/G_{12}$, and $\nu_{26}/E_2 = \nu_{62}/G_{12}$. Explicit expression for the engineering constants can be obtained from eq. (10.68)

$$\begin{aligned} E_1 &= 1 / (\beta_1 + \beta_2 + \beta_3 \cos 2\theta + \beta_4 \cos 4\theta); \\ E_2 &= 1 / (\beta_1 + \beta_2 - \beta_3 \cos 2\theta + \beta_4 \cos 4\theta); \\ \nu_{12} &= -(\beta_1 - \beta_2 - \beta_4 \cos 4\theta) / (\beta_1 + \beta_2 + \beta_3 \cos 2\theta + \beta_4 \cos 4\theta); \\ G_{12} &= 1 / (4\beta_2 - 4\beta_4 \cos 4\theta); \\ \nu_{16} &= (\beta_3 \sin 2\theta + 2\beta_4 \sin 4\theta) / (\beta_1 + \beta_2 + \beta_3 \cos 2\theta + \beta_4 \cos 4\theta); \\ \nu_{26} &= (\beta_3 \sin 2\theta - 2\beta_4 \sin 4\theta) / (\beta_1 + \beta_2 + \beta_3 \cos 2\theta + \beta_4 \cos 4\theta); \end{aligned} \quad (10.71)$$

These engineering constants can also be measured experimentally by performing the various tests depicted in fig. 10.12. The tests are similar to those discussed in section (10.5), except for the fact that the stresses are now applied at an angle θ with respect to the fibers.

Fig. 10.9 shows the variation of the modulus of elasticity E_1 as a function of the lamina angle θ . Note the precipitous drop in the modulus of elasticity when the lamina angle moves away from 0 degrees. This drop is much more pronounced than for the stiffness coefficient C_{11} .

It is important to understand the difference between the stiffness coefficient C_{11} and the engineering constant E_1 . Mathematically, these two quantities clearly are different: $C_{11} = C(1,1)$ whereas $E_1 = 1/S(1,1)$, and $C(1,1) \neq 1/S(1,1)$ since the stiffness and

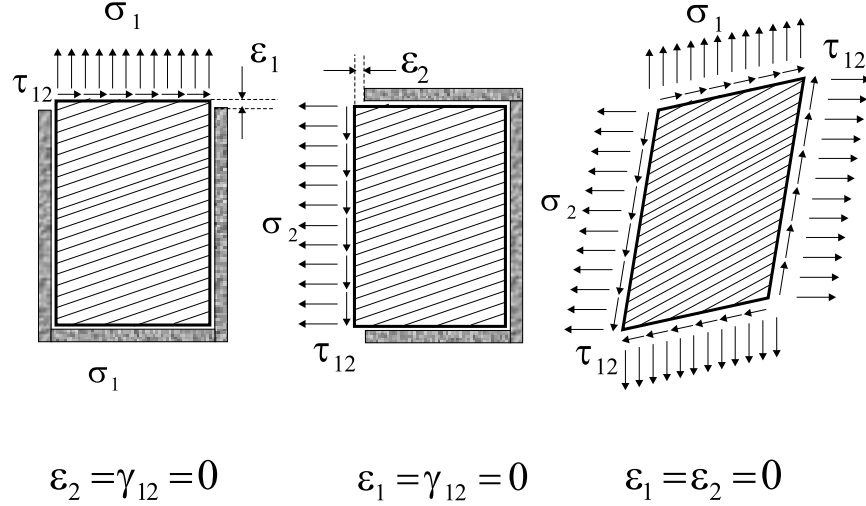


Figure 10.13: Three simple tests for the determination of the stiffness coefficients.

compliance matrices are the inverse of each other. This difference is easily understood in physical terms by looking at the tests that would allow the measurement of these quantities. Fig. 10.12 shows the tests to be performed in order to measure the engineering constants, and fig. 10.13 shows the corresponding tests to be performed in order to measure the stiffness coefficients. Focusing on the first test depicted in fig. 10.12, it is clear that a single stress component σ_{11} is applied (i.e. $\sigma_{22} = \tau_{12} = 0$). A complex state of strain results that involves ε_{11} , ε_{22} , and γ_{12} . The measurement of ε_{11} yields E_1 from the first eq. (10.64), the measurement of ε_{22} yields ν_{12} from the second eq. (10.64), and the measurement of γ_{12} yields ν_{16} from the last eq. (10.64).

In the first test depicted in fig. 10.13, a single strain component ε_{11} is applied (i.e. $\varepsilon_{22} = \gamma_{12} = 0$). A complex state of stress results that involves σ_{11} , σ_{22} , and τ_{12} . Measurement of σ_{11} , σ_{22} , and τ_{12} yield the stiffness coefficients C_{11} , C_{12} , and C_{16} from the first, second, and last eq. (10.53), respectively. Though conceptually simple, the tests depicted in fig. 10.13 are very difficult to perform in practice. The test specimen would have to be constrained so as to prevent any deformations except ε_{11} for the first test, and the stresses required to obtain that state of deformation would need to be measured. Such test is rarely performed in practice.

Considering the first test in fig. 10.13 it is clear that the stiffness coefficient C_{11} reflects the stiffness of the material when it is severely constrained since one must enforce $\varepsilon_{22} = \gamma_{12} = 0$. The effect of these constraints is to considerably stiffen the response of the material. At a 20 degree lamina angle, the stiffness coefficient C_{11} is about 130 GPa (see fig. 10.9), whereas the engineering constant E_1 is only about 50 GPa. The effect of constraining the material is clearly very important.

A similar effect is observed in fig. 10.10 which compares the stiffness coefficient C_{66} and the shearing modulus G_{12} . The stiffness coefficient increases considerably, whereas the shearing modulus rises very modestly. Both quantities, however, reach their maximum values for a 45 degree lamina angle.

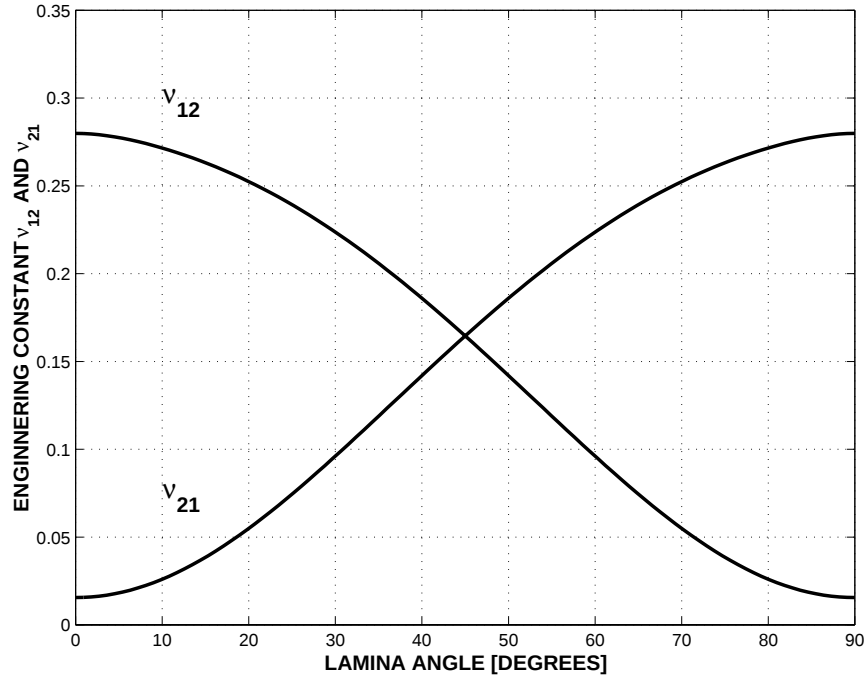


Figure 10.14: Variation of the engineering constants ν_{12} and ν_{21} with the lamina angle θ .

Fig. 10.14 shows the Poisson's ratio ν_{12} , and ν_{21} . It is interesting to note that the Poisson's ratio ν_{12} has a value of 0.28 at a 0 degree lamina angle, but a value of about only 0.02 for a 90 degree lamina angle. For most metals, Poisson's ratio is about 0.3; with composite materials, a much wider range of value is observed. Finally, fig. 10.15 shows the variation of the engineering constants ν_{16} , ν_{61} , ν_{26} , and ν_{62} as a function the lamina angle.

10.7 Strength a transversely isotropic lamina

The constitutive laws for a linear elastic, transversely isotropic material were investigated in section (10.6). The equations developed in that section express a linear relationship between stress and strain, but provide no information about failure of the material.

10.7.1 Strength of a lamina under simple loading conditions

The strength of a lamina made of transversely isotropic material can be experimentally determined by performing a series of simple tests. In numerous practical applications, this lamina will be under plane state of stress. Consider a first test where the composite is subjected to a single tensile stress σ_{11}^* in the fiber direction (i.e. $\sigma_{22}^* = \tau_{12}^* = 0$) as depicted in fig. 10.16. As the applied stress increases, a point is reached where the material fails. Let σ_{1t}^{*f} be the stress level at which failure occurs. The same test could be repeated for a compressive stress σ_{11}^* , and let σ_{1c}^{*f} be the absolute value of the compressive stress at failure. There is no reason to believe that σ_{1t}^{*f} and σ_{1c}^{*f} are, in general, equal. Therefore, the

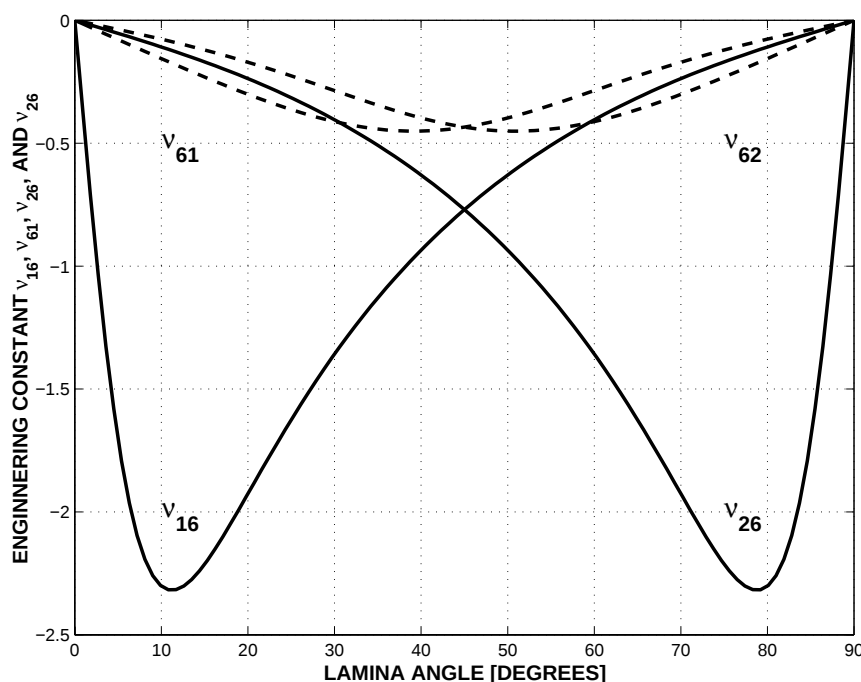


Figure 10.15: Variation of the engineering constants ν_{16} , ν_{61} , ν_{26} , and ν_{62} with the lamina angle θ .

subscripts $(.)_t$ and $(.)_c$ will be used to distinguish the tensile and compressive failure stresses, respectively.

In a second test, depicted in fig. 10.16, the lamina is subjected to a single tensile stress σ_{22}^* in the direction transverse to the fiber (i.e. $\sigma_{11}^* = \tau_{12}^* = 0$). The applied stress level that corresponds to failure of the lamina is denoted σ_{2t}^{*f} , and let σ_{2c}^{*f} be the absolute value of the compressive stress that corresponds to failure. Fig. 10.16 shows the last test to be performed: the lamina is subjected to a shearing stress τ_{12}^* ($\sigma_{11}^* = \sigma_{22}^* = 0$). Let τ_{12}^{*f} denote the level of applied shearing stress that corresponds to failure. Clearly, the failure level in shear does not depend on the sign of the shearing stress.

Though conceptually simple, the above tests can be very difficult to perform in practice. Care must be taken in the tensile tests to reinforce the ends of the test specimens that fit in the grip of the testing machine so as to avoid premature failure near the grips. Furthermore, the specimen must be long enough so that the test section is free of end effect. The test setup to measure compressive strength is far more complex because buckling of the specimen must be prevented. This can be achieved by providing lateral support of the test sample. Performing the shear test is also very complex. Subjecting a flat specimen to a state of pure shear is very difficult to achieve experimentally. A tubular specimen can of course be used, but at a far greater cost. Table 10.8 lists the failure stress levels for lamina made of different materials.

In practical design situations, the lamina might be subjected to several stress components simultaneously. Consider, for instance, a lamina subjected to stresses both along the fiber direction, and in the transverse direction. Fig. 10.17 shows the corresponding stress space,

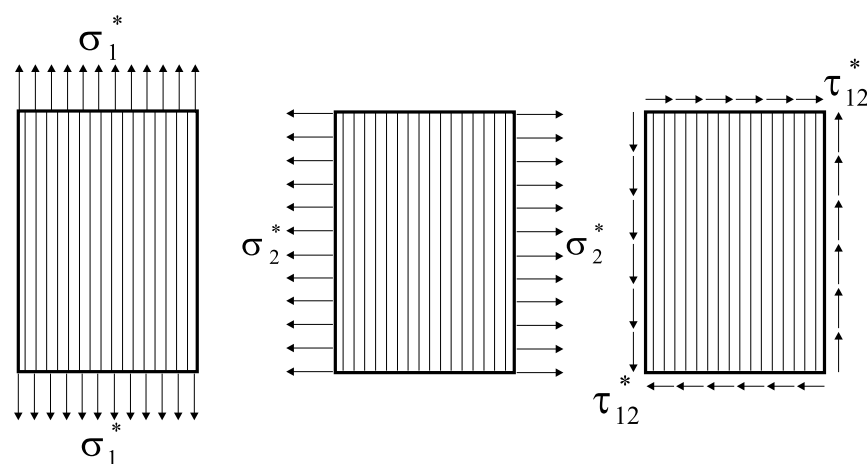


Figure 10.16: Three tests for the determination of the strength of a lamina

Material type	σ_{1t}^{*f} [MPa]	σ_{1c}^{*f} [MPa]	σ_{2t}^{*f} [MPa]	σ_{1c}^{*f} [MPa]	τ_{12}^{*f} [MPa]
Graphite/Epoxy T300/5208	1500	1500	40	240	68
Graphite/Epoxy AS/3501	1450	1450	52	205	93
Boron/Epoxy T300/5208	1260	2500	61	202	67
Scotchply 1002	1060	610	31	118	72
Kevlar 49	1400	235	12	53	34

Table 10.8: Failure stresses for lamina made of different materials.

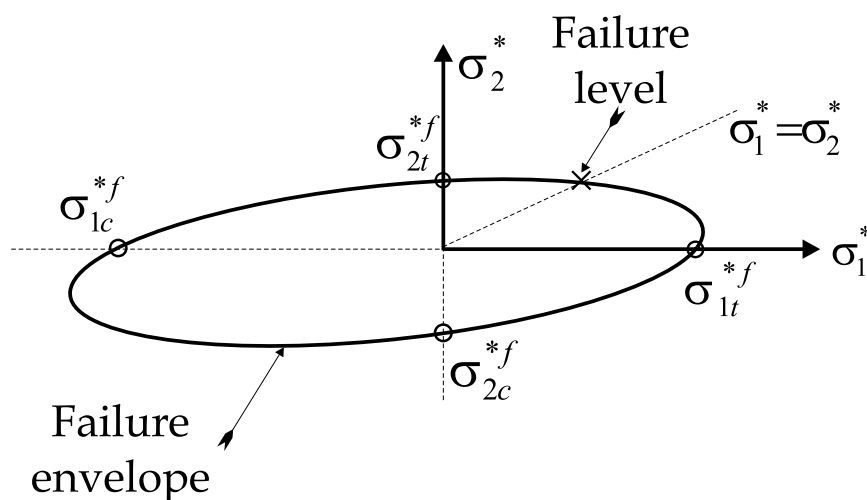


Figure 10.17: Stress space for a lamina in biaxial stress state.

and the failure stress levels σ_{1t}^{*f} , σ_{1c}^{*f} , σ_{2t}^{*f} , and σ_{2c}^{*f} which correspond to the various failure stress levels measured in the tests described earlier. Assume that equal stresses are applied in both directions simultaneously, i.e. $\sigma_{11}^* = \sigma_{22}^*$. These stress states form a 45 degree line in the stress space. As the applied stresses increase, failure will occur at a certain level. Of course, the applied stresses σ_{11}^* and σ_{22}^* could be applied in any proportion, corresponding to various radial lines emanating from the origin of the stress space. A different failure level will correspond to each radial line.

To cover all possible combinations, the failure envelope, depicted in fig. 10.17, should be known. All stress states within the failure envelope correspond to stress levels the material can sustain without failing, whereas the stress states outside the failure envelope result in failure.

Clearly, the failure envelope could be obtained experimentally by performing a large number of tests with various combinations of applied stress components σ_{11}^* , σ_{22}^* , and τ_{12}^* . This approach is not practical as it would require an enormous amount of testing to determine the failure envelope. A more desirable approach would be to determine the failure envelope based on the knowledge of a few failure stress levels such as σ_{1t}^{*f} , σ_{1c}^{*f} , σ_{2t}^{*f} , σ_{2c}^{*f} , and τ_{12}^{*f} . This can be achieved by means of a failure criterion that predicts failure under combined loads. Though many different failure criteria have been proposed, none is fully satisfactory, in the sense that their predictions are not always in very good agreement with the experimentally measured failure stresses. They are, however, widely used in preliminary design.

It is important to note that when designing with composite materials, the failure mode is often as important as the failure stress. Indeed, consider the case of a lamina subjected to a load transverse to the fibers: the lamina will fail at a very low stress level which is indicative of the the low load carrying capability of the matrix material. On the other hand, if the same lamina is subjected to a stress aligned with the fibers, it will fail at a far higher stress level which reflects the high strength of the fiber. The failure modes in the two cases are quite different: matrix failure for the former, fiber failure for the latter. Failure of the matrix due to a transverse load does not substantially decrease the ability of the lamina to

continue to carry high loads in the fiber direction, whereas with fiber failure load carrying capability is completely lost. Clearly, a matrix failure is not always a catastrophic event in contrast to fiber failure which completely eliminates any load carrying capability.

10.7.2 The Tsai-Wu failure criterion

A commonly used failure criterion is the Tsai-Wu failure criterion. This criterion states that the failure condition is reached when the combined applied stresses satisfy the following equality

$$F_{11}^* \sigma_{11}^{*2} + 2F_{12}^* \sigma_{11}^* \sigma_{22}^* + F_{22}^* \sigma_{22}^{*2} + F_{66}^* \tau_{12}^{*2} + F_1^* \sigma_{11}^* + F_2^* \sigma_{22}^* = 1, \quad (10.72)$$

where the coefficients F_{11}^* , F_{12}^* , F_{22}^* , F_{66}^* , F_1^* , and F_2^* must be determined experimentally. Note that the stress components appearing in the criterion are expressed in the fiber axis system $\mathcal{S}^*$. Consider first the test described earlier where a single stress component σ_{11}^* is applied. At failure in tension and in compression, the above equality must be satisfied, implying

$$F_{11}^* \sigma_{1t}^{*f2} + F_1^* \sigma_{1t}^{*f} = 1; \quad F_{11}^* \sigma_{1c}^{*f2} - F_1^* \sigma_{1c}^{*f} = 1. \quad (10.73)$$

The second test involves the sole σ_{22}^* stress component and yields

$$F_{22}^* \sigma_{2t}^{*f2} + F_2^* \sigma_{2t}^{*f} = 1; \quad F_{22}^* \sigma_{2c}^{*f2} - F_2^* \sigma_{2c}^{*f} = 1. \quad (10.74)$$

Finally, the last test implies:

$$F_{66}^* \tau_{12}^{*f2} = 1. \quad (10.75)$$

These five equations can readily be solved to find five of the coefficients appearing in eq. (10.72). They are

$$F_{11}^* = \frac{1}{\sigma_{1t}^{*f} \sigma_{1c}^{*f}}; \quad F_{22}^* = \frac{1}{\sigma_{2t}^{*f} \sigma_{2c}^{*f}}; \quad F_{66}^* = \frac{1}{\tau_{12}^{*f2}}; \quad (10.76)$$

$$F_1^* = \frac{\sigma_{1c}^{*f} - \sigma_{1t}^{*f}}{\sigma_{1t}^{*f} \sigma_{1c}^{*f}}; \quad F_2^* = \frac{\sigma_{2c}^{*f} - \sigma_{2t}^{*f}}{\sigma_{2t}^{*f} \sigma_{2c}^{*f}}. \quad (10.77)$$

These results are introduced in the initial statement of the failure criterion, eq. (10.72), to yield

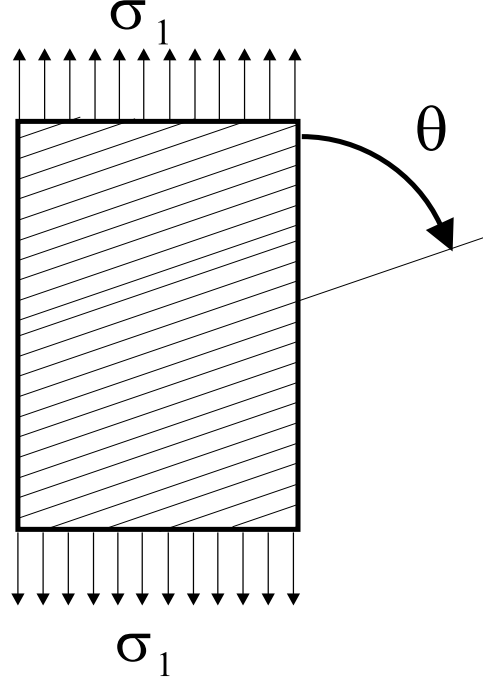
$$\bar{\sigma}_{11}^{*2} + 2\bar{F}_{12}^* \bar{\sigma}_{11}^* \bar{\sigma}_{22}^* + \bar{\sigma}_{22}^{*2} + \bar{\tau}_{12}^{*2} + \bar{F}_1^* \bar{\sigma}_{11}^* + \bar{F}_2^* \bar{\sigma}_{22}^* = 1, \quad (10.78)$$

where the following non-dimensional stress components were defined:

$$\bar{\sigma}_{11}^* = \frac{\sigma_{11}^*}{\sqrt{\sigma_{1t}^{*f} \sigma_{1c}^{*f}}}; \quad \bar{\sigma}_{22}^* = \frac{\sigma_{22}^*}{\sqrt{\sigma_{2t}^{*f} \sigma_{2c}^{*f}}}; \quad \bar{\tau}_{12}^* = \frac{\tau_{12}^*}{\tau_{12}^{*f}}; \quad (10.79)$$

as well as the following non-dimensional coefficients:

$$\bar{F}_1^* = \frac{\sigma_{1c}^{*f} - \sigma_{1t}^{*f}}{\sqrt{\sigma_{1t}^{*f} \sigma_{1c}^{*f}}}; \quad \bar{F}_2^* = \frac{\sigma_{2c}^{*f} - \sigma_{2t}^{*f}}{\sqrt{\sigma_{2t}^{*f} \sigma_{2c}^{*f}}}. \quad (10.80)$$

Figure 10.18: Strength test for a lamina at an angle θ .

The coefficient $\bar{F}_{12}^*$ is as yet undetermined. Clearly, an additional test involving a biaxial state of applied stress (i.e. a test where both σ_{11}^* and σ_{22}^* are applied simultaneously) would be required to determine this coefficient. Since such a biaxial test is very difficult to perform, the coefficient $\bar{F}_{12}^*$ is often selected by fitting the prediction of the criterion to available experimental data. $\bar{F}_{12}^* = -1/2$ has been found to provide the best fit. The final statement of the Tsai-Wu criterion becomes:

$$\bar{\sigma}_{11}^{*2} - \bar{\sigma}_{11}^* \bar{\sigma}_{22}^* + \bar{\sigma}_{22}^{*2} + \bar{\tau}_{12}^{*2} + \bar{F}_1^* \bar{\sigma}_{11}^* + \bar{F}_2^* \bar{\sigma}_{22}^* = 1. \quad (10.81)$$

As an example of application of this criterion consider the simple test shown in fig. 10.18. A single stress component σ_{11} is applied to a lamina with the fibers running at an angle θ . The stress rotation formula (1.27) yields the applied stresses in the fiber axis system $\mathcal{S}^*$ as:

$$\sigma_{11}^* = \sigma_{11} \cos^2 \theta; \quad \sigma_{22}^* = \sigma_{11} \sin^2 \theta; \quad \tau_{12}^* = -\sigma_{11} \cos \theta \sin \theta. \quad (10.82)$$

The level of applied stress that corresponds to failure satisfies the failure criterion (10.81), i.e.:

$$\begin{aligned} \sigma_{11}^2 \left[\frac{\cos^4 \theta}{\sigma_{1t}^{*f} \sigma_{1c}^{*f}} - \frac{\sin^2 \theta \cos^2 \theta}{\sqrt{\sigma_{1t}^{*f} \sigma_{1c}^{*f} \sigma_{2t}^{*f} \sigma_{2c}^{*f}}} + \frac{\sin^4 \theta}{\sigma_{2t}^{*f} \sigma_{2c}^{*f}} + \frac{\sin^2 \theta \cos^2 \theta}{\tau_{12}^{*f2}} \right] \\ + \sigma_{11} \left[\frac{\bar{F}_1^* \cos^2 \theta}{\sqrt{\sigma_{1t}^{*f} \sigma_{1c}^{*f}}} + \frac{\bar{F}_2^* \sin^2 \theta}{\sqrt{\sigma_{2t}^{*f} \sigma_{2c}^{*f}}} \right] - 1 = 0. \end{aligned}$$

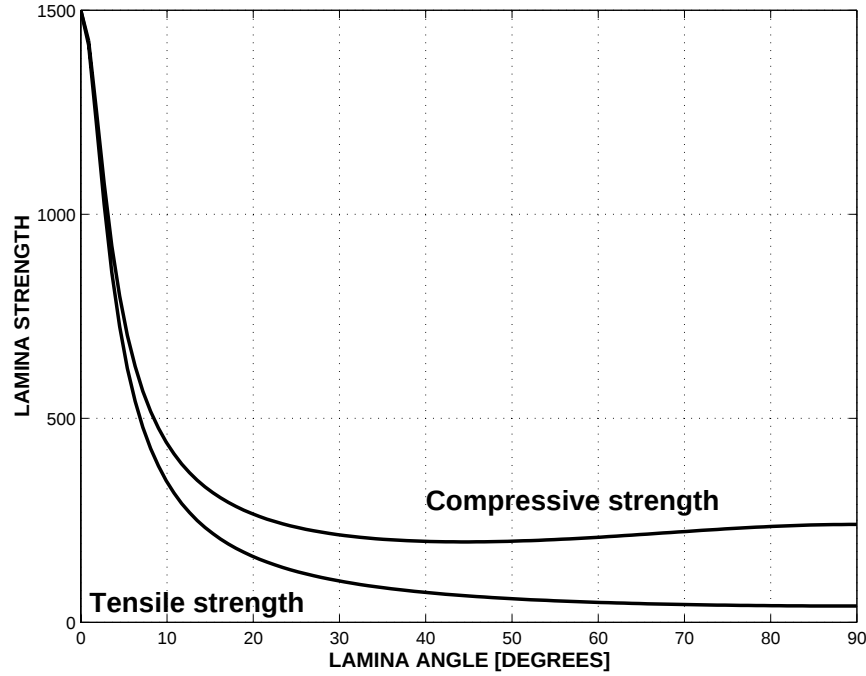


Figure 10.19: Variation of the tensile and compressive failure loads with the lamina angle θ .

This second order equation can be solved to find the failure load. The two solutions correspond to the failure loads in tension and compression. Fig. 10.19 shows the absolute value of these failure loads as a function of the lamina angle θ . Note the precipitous drop in strength as the lamina angle moves away from 0 degrees.

10.7.3 The reserve factor

The concept of reserve factor is often used in stress computations. The reserve factor R is defined as the factor by which the applied stress can be multiplied to reach failure, i.e.:

$$\sigma_{failure} = R \sigma_{applied}. \quad (10.83)$$

From this definition it follows that:

- $R = 1$ means that the applied stresses causes failure;
- $R > 1$ means that the applied stresses level is safe, i.e. it is below the failure level. A reserve factor of two means that the applied stresses can be doubled before failure occurs;
- $R < 1$ means that the applied stresses is above the failure stress.

Let σ_{11}^* , σ_{22}^* , and τ_{12}^* be the stresses applied to a lamina. By definition of the reserve factor, it follows that $R\sigma_{11}^*$, $R\sigma_{22}^*$, $R\tau_{12}^*$ is the stress level that will cause failure. Assuming

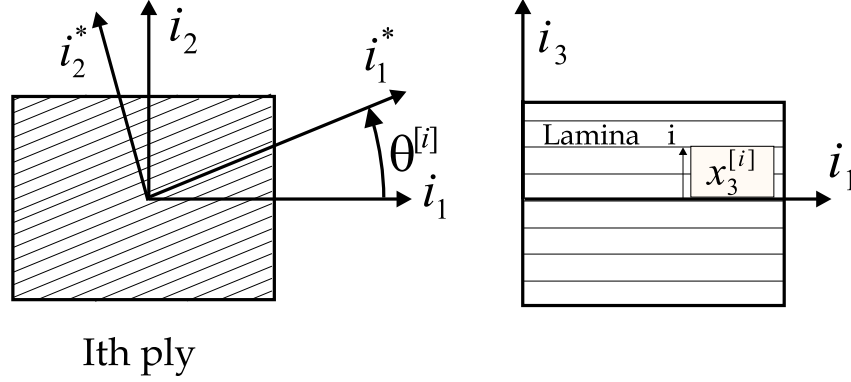


Figure 10.20: Definition of a laminated plate.

failure can be predicted by the Tsai-Wu failure criterion (10.81), the failure condition writes:

$$\frac{(R\sigma_{11}^*)^2}{\sigma_{1t}^{*f}\sigma_{1c}^{*f}} - \frac{(R\sigma_{11}^*)(R\sigma_{22}^*)}{\sqrt{\sigma_{1t}^{*f}\sigma_{1c}^{*f}\sigma_{2t}^{*f}\sigma_{2c}^{*f}}} + \frac{(R\sigma_{22}^*)^2}{\sigma_{2t}^{*f}\sigma_{2c}^{*f}} + \frac{(R\tau_{12}^*)^2}{\tau_{12}^{*f2}} + \bar{F}_1^* \frac{R\sigma_{11}^*}{\sqrt{\sigma_{1t}^{*f}\sigma_{1c}^{*f}}} + \bar{F}_2^* \frac{R\sigma_{22}^*}{\sqrt{\sigma_{2t}^{*f}\sigma_{2c}^{*f}}} - 1 = 0.$$

Introducing the non-dimensional stresses eq. (10.79), and regrouping the powers of R yields the following quadratic equation for the reserve factor:

$$(\bar{\sigma}_{11}^{*2} - \bar{\sigma}_{11}^*\bar{\sigma}_{22}^* + \bar{\sigma}_{22}^{*2} + \bar{\tau}_{12}^{*2}) R^2 + (\bar{F}_1^*\bar{\sigma}_{11}^* + \bar{F}_2^*\bar{\sigma}_{22}^*) R - 1 = 0. \quad (10.84)$$

This quadratic equation has two roots, R_1 and R_2 , respectively positive and negative. The positive root gives the failure stress level, and the negative root gives the failure stress level when the sign of the applied stresses is reversed. In general, $|R_1| \neq |R_2|$ since the failure stress level in tension and compression are, in general, different.

10.8 Laminated composite plates

The previous sections have focused on the mechanical behavior of lamina consisting of fibers all aligned in a single direction embedded in a matrix material. The mechanical properties, both stiffness and strength, were found to be highly directional. As a result, such lamina are not suitable for carrying stresses in several directions simultaneously. A possible solution to this problem is the use of laminates which consist of a number of lamina, often called layers or plies, stacked on top of each other.

Fig. 10.20 depicts such a laminated plate, consisting of N superposed plies. The axes i_1 i_2 are at the mid-plane of the plate, and i_3 is perpendicular to that plane. These axes form the reference system $\mathcal{S}$. Lamina number i has a fiber orientation angle $\theta^{[i]}$, and is located at a distance $x_3^{[i]}$ from the mid-plane. The fiber orientation angle is counted positive in the counter clockwise direction from the i_1 axis. The thickness of each lamina is $t^{[i]} = x_3^{[i+1]} - x_3^{[i]}$. The various lamina often have the same thickness, though this is not necessary.

The following notation will be used to describe a laminate:

$$[\pm 45, 0_2, 0_2, \mp 45].$$

The first layer (at the bottom of the laminate), has a fiber angle of +45 degrees, the next a -45 degree orientation. Next come two lamina with a 0 degree angle, and so on to the last layer (at the top of the laminate) which has a +45 degree angle. In many instances, the laminate possesses mid-plane symmetry, i.e. for each lamina below the mid-plane, there is a lamina of identical thickness, orientation angle, and position above the mid-plane. In such case, the top half of the laminate is a mirror image of the bottom half. This symmetry will be reflected in the notation as well: it is not necessary to repeat the description of the top half of the laminate

$$[\pm 45, 0_2, 0_2, \mp 45] = [\pm 45, 0_2]_S,$$

where the notation $[\cdot]_S$ indicates the mid-plane symmetry.

10.9 Constitutive laws for laminated composite plates

The constitutive laws for a laminated plate relate the loads applied to the laminate to the resulting deformations. These constitutive laws must be distinguished from the lamina constitutive laws which characterize an individual lamina. The goal of the present analysis is to relate the behavior of the laminate to that of the individual lamina.

Fig. 10.21 defines the in-plane forces and bending moments acting on the laminate. Let $\underline{\sigma}$ be the in-plane stresses acting on the laminate, measured in the $\mathcal{S}$ system. The in-plane forces are defined as

$$\underline{F} = \int_h \underline{\sigma} dx_3, \quad (10.85)$$

where $\int_h$ indicates an integral through the thickness h of the laminate, and $\underline{F}$ the in-plane forces $\underline{F}^T = [F_1, F_2, F_{12}]$. The in-plane forces are *improperly* called forces as they actually are forces per unit length of the laminate. The units of the in-plane forces are N/m . The bending moments are defined as:

$$\underline{M} = \int_h \underline{\sigma} x_3 dx_3, \quad (10.86)$$

where $\underline{M}^T = [M_1, M_2, M_{12}]$. Note the sign convention for the bending moments. The bending moments are *improperly* called bending moments as they actually are bending moments per unit length of the laminate. The units of the bending moments are $N.m/m$.

Let $\underline{\varepsilon}^T = [\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}]$ be the in-plane strains measured in the $\mathcal{S}$ system. The following assumption will be made

$$\underline{\varepsilon} = \underline{\varepsilon}^0 + x_3 \underline{\kappa}, \quad (10.87)$$

where $\underline{\varepsilon}^0$ are the mid-plane strains, and $\underline{\kappa}$ the curvatures. This assumption corresponds to a linear distribution of in-plane strains through the thickness of the laminate. These assumptions, called the *Kirchhoff-Love* assumptions, generalize the Euler-Bernoulli assumptions for beams. The in-plane forces, eq. (10.85), can be related to the laminate strains by introducing

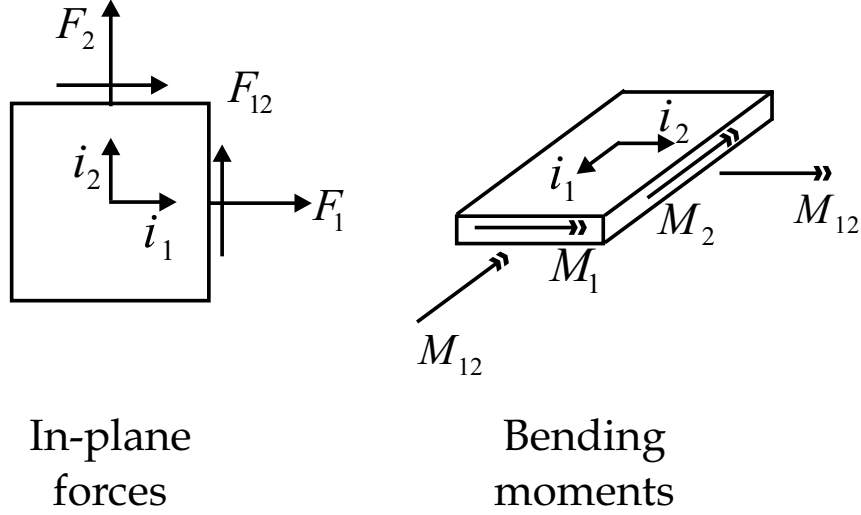


Figure 10.21: Definition of in-plane forces and bending moments on a laminate.

the constitutive laws for an individual lamina (10.53), and the assumed strain field (10.87) to find

$$\underline{F} = \int_h C \underline{\varepsilon} dx_3 = \int_h C (\underline{\varepsilon}^0 + x_3 \underline{\kappa}) dx_3 = \left[\int_h C dx_3 \right] \underline{\varepsilon}^0 + \left[\int_h C x_3 dx_3 \right] \underline{\kappa}.$$

A similar treatment for the bending moments, eq. (10.86) yields

$$\underline{M} = \int_h C \underline{\varepsilon} x_3 dx_3 = \int_h C (\underline{\varepsilon}^0 + x_3 \underline{\kappa}) x_3 dx_3 = \left[\int_h C x_3 dx_3 \right] \underline{\varepsilon}^0 + \left[\int_h C x_3^2 dx_3 \right] \underline{\kappa}.$$

The following matrices are defined:

$$A = \int_h C dx_3; \quad (10.88)$$

$$D = \int_h C x_3^2 dx_3; \quad (10.89)$$

$$B = \int_h C x_3 dx_3. \quad (10.90)$$

With the help of these matrices, the laminate constitutive laws can be written in a compact matrix form as

$$\begin{bmatrix} \underline{F} \\ \underline{M} \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}^0 \\ \underline{\kappa} \end{bmatrix}. \quad (10.91)$$

If the laminate presents mid-plane symmetry, the matrix B vanishes. Indeed, the lamina stiffness matrix C is then a symmetric function through the laminate thickness, x_3 clearly is antisymmetric, so the product $C x_3$ is antisymmetric as well and integrates to zero. As a result, for mid-plane symmetric laminates, the constitutive laws, eqs. (10.91) reduce to

$$\underline{F} = A \underline{\varepsilon}^0; \quad (10.92)$$

and

$$\underline{M} = D \underline{\kappa}. \quad (10.93)$$

The matrix A relates the in-plane forces and strains, and is called the *in-plane stiffness* matrix. The matrix D relates the bending moments and curvatures, and is called the *bending stiffness* matrix. The matrix B , called the *coupling stiffness* matrix is non zero for laminates that do not present mid-plane symmetry.

10.9.1 The in-plane stiffness matrix

Consider a laminate consisting of N plies each made of an identical material oriented at various angles. The in-plane stiffness matrix is defined by (10.88). The first term of this matrix writes:

$$A_{11} = \int_h C_{11} dx_3.$$

The integration through the thickness of the laminate can be seen as a summation of the integrals over the various plies, i.e.

$$A_{11} = \sum_{i=1}^N \int_{x_3^{[i]}}^{x_3^{[i+1]}} C_{11}^{[i]} dx_3.$$

where $C_{11}^{[i]}$ is the first component of the stiffness matrix of the i -th lamina. However, the i -th lamina can be assumed to be a homogeneous, transversely isotropic material with constant stiffness properties through its thickness. Hence,

$$A_{11} = \sum_{i=1}^N C_{11}^{[i]} \int_{x_3^{[i]}}^{x_3^{[i+1]}} dx_3.$$

Introducing the constitutive law for the lamina, eq. (10.60), and performing the integral yields

$$A_{11} = \sum_{i=1}^N (\alpha_1 + \alpha_2 + \alpha_3 \cos 2\theta^{[i]} + \alpha_4 \cos 4\theta^{[i]}) (x_3^{[i+1]} - x_3^{[i]}).$$

This expression can be written as:

$$\begin{aligned} A_{11} = & \left[\sum_{i=1}^N (x_3^{[i+1]} - x_3^{[i]}) \right] (\alpha_1 + \alpha_2) + \left[\sum_{i=1}^N (x_3^{[i+1]} - x_3^{[i]}) \cos 2\theta^{[i]} \right] \alpha_3 \\ & + \left[\sum_{i=1}^N (x_3^{[i+1]} - x_3^{[i]}) \cos 4\theta^{[i]} \right] \alpha_4. \end{aligned}$$

The first bracketed term is the total laminate thickness h . Dividing this relationship by h yields

$$\frac{A_{11}}{h} = \alpha_1 + \alpha_2 + \left[\frac{1}{h} \sum_{i=1}^N (x_3^{[i+1]} - x_3^{[i]}) \cos 2\theta^{[i]} \right] \alpha_3 + \left[\frac{1}{h} \sum_{i=1}^N (x_3^{[i+1]} - x_3^{[i]}) \cos 4\theta^{[i]} \right] \alpha_4.$$

An identical procedure can be followed to compute the various terms of the in-plane stiffness matrix. The final results can be conveniently written in terms of the following lay-up parameters:

$$\begin{aligned}\chi_1^A &= \frac{1}{h} \sum_{i=1}^N \left(x_3^{[i+1]} - x_3^{[i]} \right) \cos 2\theta^{[i]}; & \chi_2^A &= \frac{1}{h} \sum_{i=1}^N \left(x_3^{[i+1]} - x_3^{[i]} \right) \cos 4\theta^{[i]}; \\ \chi_3^A &= \frac{1}{h} \sum_{i=1}^N \left(x_3^{[i+1]} - x_3^{[i]} \right) \sin 2\theta^{[i]}; & \chi_4^A &= \frac{1}{h} \sum_{i=1}^N \left(x_3^{[i+1]} - x_3^{[i]} \right) \sin 4\theta^{[i]}.\end{aligned}\quad (10.94)$$

The components of the in-plane stiffness matrix now write:

$$\begin{aligned}A_{11}/h &= \alpha_1 + \alpha_2 + \alpha_3 \chi_1^A + \alpha_4 \chi_2^A; \\ A_{22}/h &= \alpha_1 + \alpha_2 - \alpha_3 \chi_1^A + \alpha_4 \chi_2^A; \\ A_{12}/h &= \alpha_1 - \alpha_2 - \alpha_4 \chi_2^A; \\ A_{66}/h &= \alpha_2 - \alpha_4 \chi_2^A; \\ A_{16}/h &= \frac{1}{2} \alpha_3 \chi_3^A + \alpha_4 \chi_4^A; \\ A_{26}/h &= \frac{1}{2} \alpha_3 \chi_3^A - \alpha_4 \chi_4^A;\end{aligned}\quad (10.95)$$

Note that though the laminate can comprise a large number of plies, the in-plane stiffness matrix only depends on the four lay-up parameters (10.94). Expressions (10.60) and (10.95) for a lamina, and laminate, respectively, present a striking similarity. The trigonometric function $\cos 2\theta^{[i]}$ for a lamina is replaced by the laminate lay-up parameter $(1/h) \sum_{i=1}^N \left(x_3^{[i+1]} - x_3^{[i]} \right) \cos 2\theta^{[i]}$. In other words, the lay-up parameter follows the rule of mixture, each lamina contribution $\cos 2\theta^{[i]}$ is weighted by the lamina thickness $t^{[i]} = \left(x_3^{[i+1]} - x_3^{[i]} \right)$. Since the weighting factor simply is the lamina thickness, the position of the lamina in the laminate is unimportant. The in-plane stiffness matrix is said to be *stacking sequence independent*.

It is interesting to note that the lay-up parameters are bounded by -1 and +1, like the trigonometric functions they contain, i.e.:

$$-1 \leq \chi_{1,2,3,4}^A \leq 1. \quad (10.96)$$

If the laminate possesses mid-plane symmetry, the laminate constitutive laws reduce to (10.92), which, in a reduced form become

$$\frac{F}{h} = \bar{\sigma} = \frac{A}{h} \epsilon^0, \quad (10.97)$$

where $\bar{\sigma}$ is the average stress through the thickness of the laminate. The reduced constitutive laws express a proportionality between the laminate average stresses and the mid-plane strains. Inverting eq. (10.97) yields the constitutive laws in compliance form

$$\epsilon^0 = \left[\frac{A}{h} \right]^{-1} \bar{\sigma}, \quad (10.98)$$

where $[A/h]^{-1}$ is the laminate in-plane compliance matrix. This matrix can be experimentally measured using tests similar to those described in section (10.5).

Lay-up	G [GPa]	Lay-up	G [GPa]
$[\pm 45^\circ]_3$	29.36	$[0^\circ]_{12}$	4.64
$[\pm 30^\circ]_3$	23.90	$[0^\circ_3, \pm 45^\circ]_s$	14.59
$[0^\circ_2, \pm 45^\circ]_s$	17.36	$[0^\circ_4, (\pm 45^\circ)_3]$	20.25
$[\pm 15^\circ]_3$	11.80		

Table 10.9: Tube lay-ups and corresponding shearing moduli.

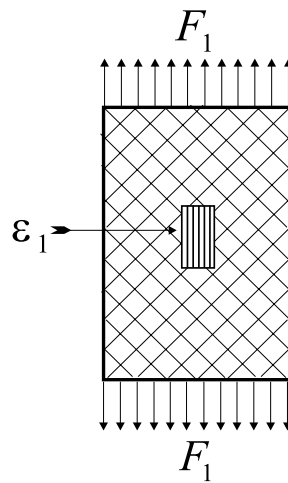


Figure 10.22: Configuration of the uniaxial test on a cross-ply laminate.

Some laminates are built in such a way that for each lamina at an angle θ , there is a corresponding lamina at an angle $-\theta$. In that case, the last two lay-up parameters χ_3^A and χ_4^A vanish. For such laminates, called *balanced laminates*, the stiffness matrix components A_{16} and A_{26} are zero. This means that balanced laminates present no coupling between extension and shearing.

10.9.2 Problems

Problem 10.1

The shearing moduli G of thin-walled tubes have been measured experimentally for tubes made of an identical material with the various lay-ups listed in table 10.9. Check how well this data correlates with classical lamination theory by plotting the shearing modulus as a function of a lay-up parameter to be determined. From the experimental data, determine the material invariants α_2 and α_4 .

Problem 10.2

Experimental measurements have been made for the following material constants of a lamina: $E_1 = 134$ GPa, $E_2 = 8.6$ GPa, and $\nu_1 = 0.29$. The shearing modulus for the lamina is to be measured from a uniaxial test on a cross-ply laminate made of that material with the lay-up

$[\pm 45^\circ]_s$. The laminate is subjected to a force F_1 and the corresponding strain ε_1 is measured, as depicted in fig. 10.22. The slope of the force F_1 versus strain ε_1 is found to be 9.5 MN/m . If the ply thickness of the material is $t_p = 125 \text{ } \mu\text{m}$, find the lamina shearing modulus E_{12} .

10.9.3 The bending stiffness matrix

The bending stiffness matrix is defined by eq. (10.89) and its first term writes

$$D_{11} = \int_h C_{11} x_3^2 dx_3.$$

The integration through the thickness of the laminate can be expressed as a summation of the integral over the various plies, then, the procedure detailed for the in-plane stiffness matrix leads to:

$$D_{11} = \sum_{i=1}^N (\alpha_1 + \alpha_2 + \alpha_3 \cos 2\theta^{[i]} + \alpha_4 \cos 4\theta^{[i]}) \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3}.$$

This expression can be written as:

$$\begin{aligned} D_{11} = & \left[\sum_{i=1}^N \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3} \right] (\alpha_1 + \alpha_2) + \left[\sum_{i=1}^N \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3} \cos 2\theta^{[i]} \right] \alpha_3 \\ & + \left[\sum_{i=1}^N \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3} \cos 4\theta^{[i]} \right] \alpha_4. \end{aligned}$$

The first bracketed term reduces to $h^3/12$. The first term of the bending stiffness matrix now writes:

$$\frac{12D_{11}}{h^3} = \alpha_1 + \alpha_2 + \left[\frac{12}{h^3} \sum_{i=1}^N \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3} \cos 2\theta^{[i]} \right] \alpha_3 + \left[\frac{12}{h^3} \sum_{i=1}^N \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3} \cos 4\theta^{[i]} \right] \alpha_4.$$

The other terms of the bending stiffness matrix can be computed similarly. The following lay-up parameters are defined:

$$\begin{aligned} \chi_1^D &= \frac{12}{h^3} \sum_{i=1}^N \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3} \cos 2\theta^{[i]}; & \chi_2^D &= \frac{12}{h^3} \sum_{i=1}^N \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3} \cos 4\theta^{[i]}; \\ \chi_3^D &= \frac{12}{h^3} \sum_{i=1}^N \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3} \sin 2\theta^{[i]}; & \chi_4^D &= \frac{12}{h^3} \sum_{i=1}^N \frac{x_3^{[i+1]3} - x_3^{[i]3}}{3} \sin 4\theta^{[i]}. \end{aligned} \quad (10.99)$$

The components of the bending stiffness matrix now write:

$$\begin{aligned} 12D_{11}/h^3 &= \alpha_1 + \alpha_2 + \alpha_3 \chi_1^D + \alpha_4 \chi_2^D; \\ 12D_{22}/h^3 &= \alpha_1 + \alpha_2 - \alpha_3 \chi_1^D + \alpha_4 \chi_2^D; \\ 12D_{12}/h^3 &= \alpha_1 - \alpha_2 & - \alpha_4 \chi_2^D; \\ 12D_{66}/h^3 &= \alpha_2 & - \alpha_4 \chi_2^D; \\ 12D_{16}/h^3 &= & \frac{1}{2} \alpha_3 \chi_3^D + \alpha_4 \chi_4^D; \\ 12D_{26}/h^3 &= & \frac{1}{2} \alpha_3 \chi_3^D - \alpha_4 \chi_4^D; \end{aligned} \quad (10.100)$$

Note the similarity between the expressions for the in-plane (10.95) and bending (10.100) stiffness matrices. The only difference is in the definitions (10.94) and (10.99) of the lay-up parameters. The bending stiffness matrix only depends on those four lay-up parameters. Once again, the lay-up parameter follows the rule of mixture: each lamina contribution $\cos 2\theta^{[i]}$ is weighted by the factor $(x_3^{[i+1]3} - x_3^{[i]3})/3$. This weighting factor explicitly depends on the position of the lamina in the stacking sequence. Hence, the bending stiffness matrix is said to be *stacking sequence dependent*. The weighting factor for the outermost plies is considerably larger than that of the lamina near the mid-plane where x_3 is nearly zero. Here again, the lay-up parameters are bounded

$$-1 \leq \chi_{1,2,3,4}^D \leq 1. \quad (10.101)$$

The stacking sequence effect can be illustrated by computing the lay-up parameter for two laminates $[0_2, \pm 45]_S$ and $[\pm 45, 0_2]_S$. All plies are assumed to have the same thickness t_p . The lay-up parameter χ_1^D of the first laminate is

$$\begin{aligned} \chi_1^D &= \frac{12}{(8t_p)^3} \frac{2}{3} \{ [(4t_p)^3 - (2t_p)^3] \cos(2 \times 0) + [(2t_p)^3 - (t_p)^3] \cos(2 \times 45) \\ &\quad + [(t_p)^3 - (0)^3] \cos(-2 \times 45) \} \\ &= \frac{1}{64} \{ 56 \times \cos(2 \times 0) + 7 \times \cos(2 \times 45) + 1 \times \cos(-2 \times 45) \} = 0.875 \end{aligned}$$

Similarly, for the second laminate:

$$\begin{aligned} \chi_1^D &= \frac{12}{(8t_p)^3} \frac{2}{3} \{ [(4t_p)^3 - (3t_p)^3] \cos(2 \times 45) + [(3t_p)^3 - (2t_p)^3] \cos(-2 \times 45) \\ &\quad + [(2t_p)^3 - (0)^3] \cos(2 \times 0) \} \\ &= \frac{1}{64} \{ 37 \times \cos(2 \times 45) + 19 \times \cos(-2 \times 45) + 8 \times \cos(2 \times 0) \} = 0.125 \end{aligned}$$

The two laminates only differ by their stacking sequence, but their lay-up parameters are very different.

If the laminate possesses mid-plane symmetry, the laminate constitutive laws reduce to (10.93), which, in a reduced form becomes

$$\frac{6}{h^2} \underline{M} = \underline{\sigma}^{max} = \frac{12D}{h^3} \frac{h}{2} \underline{\kappa}. \quad (10.102)$$

If the material were homogeneous through the thickness, the stress would be linearly distributed through the thickness of the laminate, with a maximum value $\underline{\sigma}^{max}$ at the top of the laminate. On the other hand, due to the assumed linearity of the in-plane strains (10.87) through the thickness of the laminate, the strains at the top and at the bottom of the laminate are

$$\underline{\epsilon}^{top} = \underline{\epsilon}^0 + \frac{h}{2} \underline{\kappa}; \quad \underline{\epsilon}^{bottom} = \underline{\epsilon}^0 - \frac{h}{2} \underline{\kappa}. \quad (10.103)$$

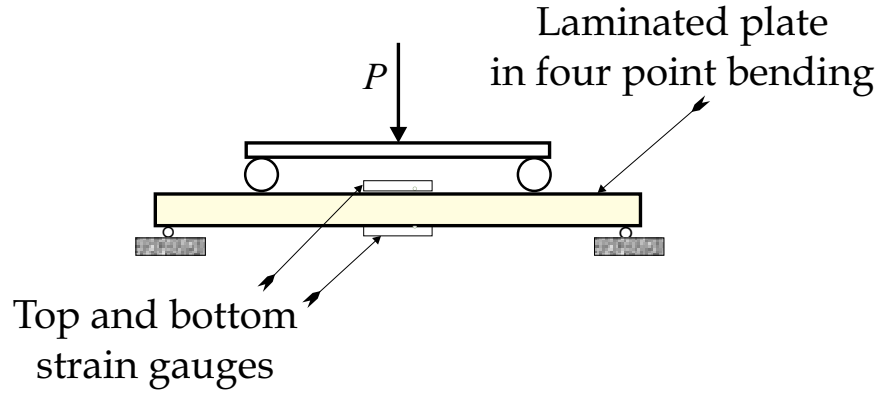


Figure 10.23: Test set-up for the measurement of the bending stiffness matrix components.

When the laminate is in pure bending $\underline{\varepsilon}_{11}^{top} = \underline{\varepsilon}_{11}^{bottom} = h\underline{\kappa}/2$. The reduced constitutive laws now write:

$$\underline{\sigma}^{max} = \frac{12D}{h^3} \underline{\varepsilon}^{top}. \quad (10.104)$$

These reduced constitutive laws express a proportionality between the stresses and strains at the top of the laminate for an *equivalent*, homogeneous material. Inverting eq. (10.102) yields the constitutive laws in compliance form as:

$$\frac{h}{2} \underline{\kappa} = \left[\frac{12D}{h^3} \right]^{-1} \frac{6}{h^2} \underline{M} \quad (10.105)$$

where $[12D/h^3]^{-1}$ is the laminate bending compliance matrix which can be experimentally measured by performing the tests depicted in fig. 10.23. A four point bending test set-up is used to put the laminate in pure bending. Fig. 10.23 depicts an applied bending moment M_1 , and the resulting curvature is measured by means of strain gauges as $\kappa_1 = (\varepsilon^{top} - \varepsilon^{bottom})/h$.

It is interesting to note that a *balanced laminate* construction does not imply the vanishing of the bending matrix components D_{16} and D_{26} . Indeed the lay-up parameters χ_3^D and χ_4^D depend on the orientation angle, but also on the stacking sequence. In other words, the contributions of two adjacent plies, at angles θ and $-\theta$, respectively, do not cancel because the two plies are at a different position in the stacking sequence. However, it often happens that the bending stiffness matrix components D_{16} and D_{26} are very small compared to the others. For such laminates, called *specially orthotropic laminates*, no coupling exists between bending and twisting.

10.9.4 Problems

Problem 10.3

The strain states at the upper and lower surfaces of a laminated composite plate have been measured by means of strain gauges. Fig. 10.24 depicts the arrangement of the three gauges on the upper surface of the plate, noted ε_1^u , ε_2^u , and ε_3^u . The corresponding gauges on the lower surface of the

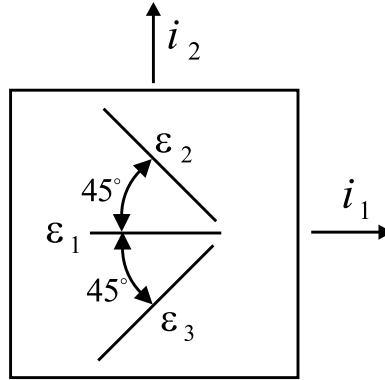


Figure 10.24: Configuration of the three strain gauges at the upper surface of the plate.

plate are ε_1^l , ε_2^l , and ε_3^l . The following strains were measured

$$\begin{aligned} \varepsilon_1^u &= 3561 \mu & \varepsilon_2^u &= 2804 \mu & \varepsilon_3^u &= 5011 \mu \\ \varepsilon_1^l &= 1069 \mu & \varepsilon_2^l &= -464.1 \mu & \varepsilon_3^l &= -1603 \mu \end{aligned}$$

The material properties are as follows: $E_1 = 140 \text{ GPa}$, $E_2 = 9.0 \text{ GPa}$, $E_{12} = 6.0 \text{ GPa}$, $\nu_1 = 0.3$. The ply thickness is $t_p = 125 \mu\text{m}$ and the lay-up is $[0_2^\circ, \pm 45^\circ]_s$.

1. Compute the axial forces $\underline{F}$ and bending moments $\underline{M}$ in the laminate.
2. Compute the stresses in the 45° ply, in the axes aligned with the fiber.

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